that makes for righteousness"; faith that this life does not end all; faith that our lives today prepare for our lives somewhere beyond.

Tangible Immortality. Out of the manifold characteristics of the mind of youth, one of the most interesting is best described by the word "opinionatedness"—a word that I grant should be in the *Index expurgatorius*. All normal individuals reach this milestone, most of them pass it, but some find their development arrested at this point, and here they bind themselves for life. In particular, they absorb (I will not say develop) the idea that immortality is an idle word of an idle faith, a part of the "opiate of the proletariat," as our most modern autocracy has phrased it.

One thing that mathematics early imparts, unless hindered from so doing, is the idea that here, at last, is an immortality that is seemingly tangible,—the immortality of a mathematical law. The student of algebra, for example, may well question the use of the traditional curriculum, but when he finds the value of $(a + b)^2$ he has come in contact with an eternal law. The laws of the Medes and Persians, unchangeable though they were thought to be, have all perished; the canons that bound Egyptian activities for thousands of years exist only in the ancient records, preserved in our museums of antiquity; the laws of Rome, which at one time dominated the legal world, have given place to modern codes; and the laws that we make today are certain to be changed tomorrow. But in the midst of all these changes it has ever been true, it is true today, it shall be true in all the future of this earth, and it is equally true throughout the universe whether in the algebra of Flatland or in that of the space in which we live, that $(a + b)^2 = a^2 + 2ab + b^2$.

We may change the symbols,—they are temporary expedients to convey the idea; we may speak in different tongues,—they are local expedients to convey thought; but it is inconceivable to us that the relation which the formula expresses should not be true always and everywhere,—a tangible symbol of the immortality of law.

What I learned in chemistry, as a boy, seemed true at the time, but much of it today is known to be false. What I learned of molecular physics seems at the present time like children's stories, interesting but puerile. What we learn in history may be true in some degree, but is certain to be false in many particulars. So we may run the gamut of learning, and nowhere, save in mathematics alone, do we find that which stands as a tangible symbol of the immortality of law, true "yesterday, today, and forever."

But does the teacher make this known to the student? Does the student come to feel the significance of this fact,—this fact so full of awe to the normal mind, this evidence of immortality that never comes to his consciousness until he meets it in mathematics? I do not know, nor do I know how much else that is great, that has tremendous significance, is taught or is not taught in science, in letters, in history, or in art. I only know that mathematics can do this thing; that it can (and it should) give, to the degree that the pupil is able to receive it, the idea that before the world was created, before our solar system was formed,

and after our system shall cease to be, the every-day laws of mathematics stood and shall stand for immortal truth,—for laws that are divine in their infinite endurance. It is not necessary, it is not desirable, that we should preach these things in the halls of learning; but it is essential that we should feel their significance. This done and all the rest "shall be added unto you." Stated in another way, the immortality of law means that we come in touch with the invariant. The tyro in mathematics comes early upon the invariant properties of a figure as seen in the theory of elementary projection. In a wider sense, however, all geometry is a science of invariance. We prove a law for a general plane triangle and it never varies, whatever we do to the figure. If we prove that $a^2 = b^2 + c^2 - 2bc \cos A$, then, however A may change, the law itself will never vary. In it the pupil comes into touch with the unchangeable, with the absolute.

It is the same with all other laws of geometry. In any convex polyhedron, whatever its shape, the law remains that the number of faces plus the number of vertices is equal to the number of edges increased by two.

"Change and decay in all around I see," but the established properties of a general geometric figure, in our space, are as unchangeable in that space as divinity itself. Stated in still another way, the immortality of law and the invariability of mathematical principles mean the eternity of mathematics. To come into relation with a science which was illustrated by the spiral nebulae before our solar system was formed, which only now reveals to us those laws of crystals which were in operation long before life appeared upon the earth, and which is also entirely independent of matter, so that if we could imagine the universe destroyed absolutely, the laws would still be true,—to come into relation with such a science makes real to us, as no other discipline in our curriculum can possibly do, the ideal of truth eternal.

Our Infinitesimal Nature. I know of nothing which acts as such a powerful antidote to that which I ventured to call "opinionatedness," as a study of mathematics. To know that the light from solar systems far larger than our own has been thousands of years in reaching us, gives us an idea of our infinitesimal nature, in comparison with space about us, that can come only with a study of the science that it is ours to teach. A bacillus in our veins, so small as to be invisible through a powerful microscope, is a giant compared with ourselves in our relation to this space in which we live. Our doubts, our beliefs, our hopes, our fears are all so trivial, so infinitesimal, so like a lost electron in our solar system, as compared with our relative importance in the universe as revealed to us by the calculations which mathematics brings to bear upon the great problem! Cowper wrote well when he put in verse the words,

> "God never meant that man should scale the heavens
> By strides of human wisdom."

and even the mathematics of youth confirms the thought.

With our feeble voices we cry out that we are certain that this life ends our existence, but mathematics shows us the truth of Voltaire's words, in speaking of human opinion of the significance of life,—

"Doubt is not a pleasant condition, but certainty is an absurd one."

Our feeble voices join in a protest that we cannot understand God. Again the "old man of Ferney," condemned by so many as an atheist, speaks out and says,

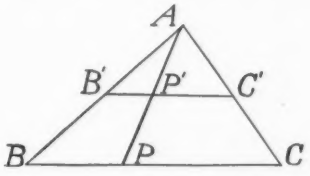
"My reason tells me that God exists; but it also tells me that I cannot know what He is."

The infinitesimal still cries out that it cannot understand what our soul is, and again the sage speaks:

"Is it indeed likely that we should know what our soul is, when we could form no idea of light if we had the misfortune to be born blind?"

We agree that spaces of higher dimensions than the one in which we think we live can easily be conceived by analogy, and we agree without question to the paradoxes which we meet in the study of infinity, and yet we feel that it shows our great wisdom, or perhaps our boldness, if we deny the soul an existence. Strange, that in algebra we accept without the slightest question the idea of the permanence of law, but that our little natures should so often boast that we deny the permanence of the soul!

Contact with the Infinite. One of the impressive experiences which comes to the devotees of our science is the continual contact with the infinite,—an experience which is inspiring beyond words to express,—and sometimes as discouraging. To take an illustration that is a mere commonplace to those who have even visited the borderlands of our science, suppose that a line segment $B'C'$ joins the mid-points of the sides AB and AC of the triangle ABC, and that the straight line $AP'P$ connects the vertex A with points P' and P on $B'C'$ and BC respectively. It is then evident that, in this construction, to any point P on BC there is one point and only one point P' on $B'C'$ that corresponds to it, and conversely. Like it or not as we please, there is only one possible conclusion, namely, that there is a one-to-one correspondence between the points on $B'C'$ and the points on BC; or, taking the usual definition of equality, that there are precisely as many points on $B'C'$ as there are on BC, which is twice as long. But similarly, the number of points on $B'C'$ can be shown to be the same as the number on any part of BC, and so a part (not of a line, but of a group of points) is equal to the whole. To adopt a paradox of old Tertullian, "Certum est quia impossibile est."



Against such a conclusion our little minds revolt; we have been taught, or we have empirically learned, that the part is less than the whole,—as it really is in dealing with the finite. But when we come to deal with the infinite, our little, narrow, finite laws break down; they are even more feeble than the beliefs and experiences of the child, when viewed by the eyes of a mature man. We must face the inevitable, that in the domain of the infinite, the part may indeed be equal to the whole, in spite of the childish beliefs of our finite minds. The mathe-

matician continually meets this necessary conclusion; it is a commonplace; to him the laws of the finite give way without question when he enters the domain of the infinite. In all the finite, indeed, he sees the infinite. He knows that there is a one-to-one correspondence between the points in a sphere and those in the entire universe about him, and between the number of vibrating points in his brain and the number in all space. In other words, within the brain of each of us, there is a point that corresponds to any given point in the universe, say one on the surface of Neptune. If we take another point, say the center of a fixed star, there is one special point, and only one, within each brain that also corresponds to that. If we move the point in the fixed star ever so slightly, we move the point in the brain so that it shall continue to correspond to it. Does all this signify anything to us? I certainly do not know. Does it mean that the planets have their influence upon us? I do not know, but with my finite mind I am led to say I do not think so. Does this mathematico-physical relation of our brains to all the universe about us have any deeper significance? I do not know. But one thing I do know, that thoughts like these give new meaning to the words, "For, behold, the Kingdom of God is within you."

Do such ideas signify that it is ours to preach? Shall mathematics become a medium for religious instruction? Do we not lower religious belief when we link it to the certainties of cold mathematics? With respect to each of these questions I would reply, "I do not think so." With respect to the last one, I see no difference in the sanctity of truth, whether the truth be taught in the books of Euclid, in the holy books of the East, or in the Christian Church;—in the *significance* of truth,—yes; in the *sanctity* of truth,—no.

Our Impotence in Relation to the Eternal. I know of no other branch of learning that makes so clear to us our impotence in relation to the Eternal. We fail today in a problem in chemistry, but we feel that we may succeed tomorrow; we are at a loss to know how to overcome a certain difficulty in physics, but we have confidence that someone, sometime, somehow, will overcome it; we do not attain to the success we had hoped for in the painting of a certain picture, but we "carry on" in the hope of bettering our work from day to day; and so in mathematics, we fail but we persevere. But there come times in mathematics when we fail and know that we must fail, because we come in conflict with the Eternal. At first sight we say that we can construct a seven-edged polyhedron. We fail, we seek out the reason, and we find that we are combatting the everlasting truth of which I have spoken, the truth that $f + v = e + 2$. Protest as we will, we are powerless when we combat the Eternal.

Physical Permanence. Our mathematics also comes to the aid of science and assists in the proof that force is not lost, and shows us that even a thought generates a wave which has its eternal influence. I like the ancient theory, and I know of no reason why it should not be a fact, that the very walls and arches of the venerable cathedrals of the old world, which have heard the daily chants of the priests for many centuries, have come to vibrate in unison therewith, so

that the same singers, if transported to another spot, fail to produce the same sonorous, musical effect. Why, then, should not the thoughts of others influence us, as the telepathists affirm, and why should not our body of thought be permanent in space, even from a materialistic point of view?

Mathematics is such a science of harmony! The universe is such a science of harmony! The "music of the spheres" meant precisely this in the ancient philosophy;—and how out of all harmony with what mathematics reveals to us is the theory of annihilation, even, as I have said, from the materialistic standpoint!

To quote again from Voltaire, whose name is anathema to so many ignorant religionists:

"All nature cries aloud that He *does* exist; that there *is* a supreme intelligence, an immense power, an admirable order, and that everything teaches us our own dependence upon it."

Spoken like a philosopher, but also like the mathematician that he was; for Voltaire was a great student of Newtonian philosophy and did more than any man in his century to make the teachings of Newton known to the general intellectual element of France. That Voltaire, the bitter antagonist of fraud and sham, in the Church as well as in the State, should honestly confess his faith in this manner need not surprise us. It shows his greatness. Cicero told the world two thousand years ago that the greatest thinkers always have such faith:—

"There is, I know not how, in the minds of men a certain presage, as it were, of a future existence; and this takes the deepest root and is most discoverable in the greatest geniuses and the most exalted souls."

The Drama of Space. What a science is ours that raises the curtain on the drama of space! That shows us a finite space in which all bodies are at rest until acted upon by some external force; and then the space of the infinitesimal, in which a radically different code of laws obtains,—where everything (molecules, atoms, electrons, and very likely sub-electrons) is an automaton, self moved and never resting. John Burroughs has expressed this idea, adding:

"When we reach the astronomic world, or the sidereal universe, we find the same condition that prevails in the world of the infinitely little: perpetual motion goes on, friction is abolished, and nothing is at rest. . . . Height and depth, upper and under, east and west, north and south, weight and inertia, as we experience them, have vanished. There are no boundaries, no ending and no beginning, no center and no circumference; the infinite cannot have any of these."

What a science is ours, moreover, that reveals to the youth the secret of indirect measurement! That shows him how the distances to the stars are found! That opens to him the moving picture of the universe! That reveals the world of the electronic bodies as the reciprocal of the world of the sidereal bodies! And who can measure the influence of this revelation upon the soul of one who is standing upon the threshold of young manhood or young womanhood?

Scientific Religion. Religion is generally felt to be unscientific. One of the world's great authorities, Professor Harald Höffding of Copenhagen, in his *Philosophy of Religion*, speaks of it as that state of mind

"in which feeling and need, fear and hope, enthusiasm and surrender play a greater part than do meditation and inquiry, and in which intuition and imagination have the mastery over investigation and reflection."

I cannot believe that this should be the case with respect to the great basal facts of religion, although it is so traditionally, at least in our sectarian doctrines. We lay down certain postulates in geometry; they may, as in the case of parallels, be true or false; mathematics simply says, "If *A*, then *B*,"—if these postulates are sound, then these conclusions are true. I have often wondered mildly why religion did not do the same, postulating certain statements and then proceeding precisely as in mathematics,—"If *A*, then *B*." The postulates, like those of Euclid, might be true or false, but the deductions would be absolute. I have no idea as to what postulates would be assumed; I simply know that it seems entirely scientific to assume at least a few that are reasonable. We do not, in elementary mathematics, feel that our postulates must be independent or that they must cover all possible needs; we simply assume what seems to be necessary for the mind of youth, and on that we build.

When I think, in a kind of indefinite fashion, of what my mind postulates with respect to some of the larger features of mathematics, my thought runs to something like this:

1. The Infinite exists.
2. Immortal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.
4. The existence of hyperspace is entirely reasonable.
5. No factor is ever lost.
6. Time may be a closed curve.

Such a list of postulates might easily be put into theological language as well, and might be extended when necessary. For example, the theologian might phrase these same postulates like this:

1. God exists.
2. God's laws exist.
3. God's laws are entirely different from ours.
4. There are spaces beyond ours.
5. The soul exists and is eternal.
6. God looks at time as a whole.

After making out his list of postulates, the theologian might formulate his definitions. It would be as easy, I should think, to make an attempt to define God, heaven, angel, and miracle as it is to make an attempt to define infinity, hyperspace, straight line, solid angle, and dozens of other terms that we use in mathematics; or, if precise definition were found unnecessary, as is the case with these mathematical terms, at least some reasonable limitations would be in order.

Given the postulates and the definitions, I see no reason why a perfectly rigorous set of propositions should not be erected and religion put on a cold, scientific foundation. I should not wish to see this done; I think it would be

about as sensible as to build up a scientific, deductive system of love or beauty; but what I mean to say is that if religion is unscientific, it is partly because the world wishes it to be so. Love would not be any more potent, or more real, or more beautiful if we formulated a set of postulates and deduced a series of propositions relating to it; and the same may be said of religion.

Duality of Mathematics and Religion. Schopenhauer's duality between Time and Space is well known. Time is homogeneous, for example; it is a continuum, and no part is separated from any other part by something which is not time. The dual proposition for Euclidean space is simply formed by a substitution of "space" for "time."

We need not be surprised at this dualism. Time has often been called a fourth dimension, and it harmonizes with our three dimensional, Euclidean space to think of time alone as a space of one dimension. The two united gave to Sir William Rowan Hamilton the science of quaternions, and now they unite again to give birth to one feature of the Minkowski-Einstein hypothesis.

How much of Schopenhauer's duality is real we cannot say, because we cannot say how much of time and space are real. If we leave the domain of Euclidean space and think of ourselves as living in a space that curves through a fourth dimension, as a spherical two-dimensional space curves through a third dimension, then this space becomes limited, as Einstein suggests, and we are led to consider other three-dimensional spaces like ours,—all quasi-finite, and the Schopenhauer dualism automatically changes.

Similarly, when we consider the duality between mathematics and religion, we have nothing positive. Change the space in which we live, make time a fourth dimension, let our new universe curve through a fifth dimension, and the details of the parallelism would necessarily change. Grant, however, certain postulates, derived empirically like the postulates of mathematics, or, as we say, derived from common sense,—grant these, and some such parallelism as the following is suggested to the mathematical mind:

MATHEMATICS.	RELIGION.
1. The infinite exists.	1. God exists.
2. Eternal laws exist.	2. Eternal laws exist.
3. The laws relating to finite magnitudes do not hold respecting the infinitely large or the infinitely small.	3. God's laws are so different from ours as to be absolutely non-understandable by us.
4. The existence of hyperspace is entirely reasonable.	4. The existence of a heaven, with gradations, is entirely reasonable.
5. No factor is ever lost.	5. The soul is eternal.
6. Time may be a closed curve.	6. God looks at time as a whole.
7. Time may be a fourth dimension.	7. In the next world, the direction of time may actually be seen.

8. Positive infinity may physically coincide with negative infinity, if lines curve through four-space.	8. In God's sight the infinite past and the infinite future are the same.
9. A Flatlander has enough of the third dimension in his being to give him some feeling of that dimension; and so this may explain the fact that we have some feeling of the fourth dimension.	9. The human soul has enough of the divine within it to have some feeling of the reality of divinity and of the world beyond.
10. Mathematics is a vast storehouse of the discoveries of the human intellect. We cannot afford to discard this material.	10. Religion is a vast storehouse of the discoveries of the human spirit. We cannot afford to discard this material.
11. It is not necessary that the solution of a problem, by limited means,—say the trisection of an angle,—should be found in order that we may feel certain that the problem can be solved by *some* means.	11. It is not necessary that the solution of the problem of religion, by our limited human means, should be found in order that we may feel certain that the problem can be solved by *some* means.
12. Every term in an infinite sequence is in a small way a part of infinity.	12. Lucretius spoke wisely when he said, "Everyone is in a small way the image of God."

Conclusion. And what is the conclusion? Does mathematics make a man religious? Does it give him a basis for ethics? Will the individual love his fellow man more certainly because of the square on the hypotenuse?—Such questions are trivial; they are food for the youthful paragrapher. Mathematics makes no such claim. What we may safely assert, however, is this,—that mathematics increases the faith of a man who has faith; that it shows him his finite nature with respect to the Infinite; that it puts him in touch with immortality in the form of mathematical laws that are eternal; and that it shows him the futility of setting up his childish arrogance of disbelief in that which he cannot see.

And if this be the case, then what is the duty of teachers of our science? To preach?—that should be the last thought. The greatest sermons are preached in silence. The most ancient religions that we have, if there be more than one fundamental religion, have always recognized this fact. And so it must be with us,—that we should teach "the science venerable" not merely for its technique; not solely for this little group of laws or that; not only for a body of unrelated propositions or for some examination set by the schools; but that we should teach it primarily for the beauty of the discipline, for "the music of the spheres," and for the faith that it gives in truth, in eternal law, in the Infinite, and in the reality of the imaginary; and for the feeling of humility that results from our comparison of the laws within our reach and those which obtain in the transfinite

domain. With such a spirit to guide us, what teachers we would be!—whether of those who are standing on the threshold, of those who are passing through the realms of mystery that lead to manhood and womanhood, of those of mature years, or of those who, as the ancients were wont to say, "number their years upon their right hand." For then, unconsciously but none the less surely, would we prove the words of a seer among poets:

"And Reason now, through number,
time, and space
Darts the keen lustre of her serious eye;
And learns, from facts compared,
the laws to trace
Whose long procession leads to Deity."

MATHEMATICAL ASSOCIATION OF AMERICA.

The sixth annual meeting of the Association is to be held at Toronto December 29–30, 1921. The American Association for the Advancement of Science and the American Mathematical Society are to hold their annual meetings there during the same week.

The following 49 persons and 1 institution, on applications duly certified, have been elected to membership.

G. H. ALBRIGHT, A.M. (Harvard). Prof., Colorado College, Colorado Springs, Colo.
THURMAN ANDREW, B.S. (W. Va. Wesleyan). Teacher, Wesleyan College, Buckhannon, W. Va.
W. F. C. ARNDT, Ph.D. (Göttingen). Prof., Grey Univ. Coll., Bloemfontein, South Africa.
E. A. BAILEY, A.B. (Georgia). Instr., U. S. Naval Academy, Annapolis, Md.
W. W. R. BALL, M.A. Trinity College, Cambridge, Eng.
E. B. BEATTY, A.M. (California). Asso. prof., Oregon Agric. Coll., Corvallis, Ore.
B. C. BELLAMY, B.S.C.E. (Wyoming). Civil Engineer, Laramie, Wyo.
O. J. BOND, LL.D. (South Carolina). Supt., The Citadel, Milit. Coll. of South Carolina, Charleston, S. C.
B. H. CAMP, Ph.D. (Yale). Prof., Wesleyan Univ., Middletown, Conn.
C. H. CHEPMELL, Major, Sussex, Eng.
CARRIE A. DAY (Fredonia State Normal). Teacher, H. S., Smethport, Pa.
L. G. DUPASQUIER, Ph.D. (Zurich). Prof. ord., Univ. Neuchatel, Switzerland.
H. W. ENGLISH. Master in Chancery, Jacksonville, Ill.
HENRI FEHR, Docteur ès sciences (Geneva). Prof., Univ. of Geneva, Switzerland.
PHILIP FITCH, A.B. (Colorado Coll.). Instr., physics, N. Side H. S., Denver, Colo.
A. R. FORSYTH, Sc.D., LL.D., Math.D. Chief prof. of math., Imperial Coll. of Sc. and Tech., S. Kensington, London, Eng.
TORWALD FREDERICKSEN, Ph.B. (Chicago). Instr., Colgate Univ., Hamilton, N. Y.
GUIDO FUBINI. Prof., Univ. and Polytech. School of Turin, Turin, Italy.
J. C. FUNK, A.M. (Columbia). Head of dept. of math., H. S. and Junior Coll., Santa Maria, Calif.
C. L. GARNER, B.E. (N. C. State Coll.). Hydrogr. and geod. engr., U. S. Coast and Geod. Survey, Washington, D. C.
R. A. P. GÉRARDIN, Editor *Sphinx-Œdipe*, Nancy,France.
J. HADAMARD, Prof., Ecole Polytech. and Coll. de France, Paris, France.
G. H. HARDY, Savilian Prof. of Geom., Oxford Univ., Oxford, Eng.
R. R. HAUN, A.M. (Vanderbilt). Prof., Ashland Coll., Ashland, Ohio.
I. O. HORSFALL, A.B. (Utah). Head of dept. of math., Latter Day Saints H. S., Salt Lake City, Utah.
L. A. HOWLAND, Ph.D. (Munich). Prof., Wesleyan Univ., Middletown, Conn.

H. M. JEFFERS, Ph.D. (California). Instr. in astr., Iowa State Univ., Iowa City, Iowa.
R. P. JOHNSON, A.M. (Virginia; Harvard). Instr., U. S. Naval Academy, Annapolis, Md.
H. V. KNORR, A.B. (Susquehanna Univ.). Head of dept. of math., Central Wesleyan College, Warrenton, Mo.
ANNA D. LEWIS, Ph.D. (Carleton). Prof. of math. and physics, Lake Erie College, Painesville, Ohio.
C. F. LEWIS, A.B. (Denver). Instr., Kansas State Agric. Coll., Manhattan, Kansas.
FLORENCE LONG, M.S. (Illinois). Asst. prof., Earlham College, Earlham, Ind.
E. S. MANSON, Jr., M.S. (Mass. Inst. of Tech.). Asso. prof. astr., Ohio State University, Columbus, Ohio.
W. J. MARTYN, M.A. (New Zealand). Math. Master, Otago Boys H. S., Dunedin, N. Z.
NINA M. MCLATCHEY, A.B. (Washburn). Instr., Univ. of Kansas, Lawrence, Kans.
A. A. MCSWEENY, A.M. (Montana). Asst. prof., math. and astr., Albion Coll., Albion, Mich.
YOSHIO MIKAMI. Research assoc., Imperial Acad., Tokyo, Japan.
A. L. MILLER, Ph.D. (Harvard). Brookline, Mass.
N. B. MITRA, M.A. (Calcutta). Prof., Ewing Christian Coll., Allahabad City, India.
W. B. PIETENPOL, Ph.D. (Wisconsin). Asso. prof., physics, Univ. of Colorado, Boulder, Colo.
C. J. REFS, A.B. (Franklin and Marshall). Asst. prof., Delaware College, Newark, Del.
L. P. RIPPY, Ph.B. (Elon College). Elon College, N. C.
JOSEPH SEIDLIN, A.M. (Columbia). Prof., physics, Alfred Univ., Alfred, N. Y.
C. J. STOWELL, Ph.D. in Econ. (Illinois). Prof., math. and physics, McKendree Coll., Lebanon, Ill.
MARY S. TAYLOR, A.B. (Cornell). Teacher, N. Y. Training School for Teachers, New York, N. Y.
W. A. TITSWORTH, A.M. (Alfred), M.S. (Wisconsin). Prof., Alfred Coll., Alfred, N. Y.
H. G. TITT, A.M. (Michigan). Dean and prof. of math., Fairmount Coll., Wichita, Kans.
E. B. VAN VLECK, Ph.D. (Göttingen). Prof., Univ. of Wisconsin, Madison, Wis.
B. F. WALTON. Instr., Randolph-Macon Coll., Ashland, Va.

HARVARD COLLEGE, Cambridge, Mass., to institutional membership.

W. D. CAIRNS,
Secretary-Treasurer.

June, 1921.

PROPOSED AMENDMENT TO THE BY-LAWS OF THE ASSOCIATION.

At the Wellesley meeting of the Board of Trustees of the Association, the following amendment to the By-Laws was recommended to the Association for adoption at the next annual meeting, embodying the report of a committee appointed by the Trustees from among the actuarial members of the Association. The committee (Professors E. L. Dodd, chairman, J. W. Glover, and H. L. Rietz) express their judgment that this is an actuarial equivalent for the four dollar membership.

"On the payment of $100 - x$ dollars in one payment, where x is the number of years in the member's age, a member may become a life member and shall thereafter be exempt from all annual dues. A separate account shall be kept of life membership dues, this account to be credited at the end of each interest period with interest earned, and to be charged at the beginning of each year with \$3.75 [1] for each surviving life member."

This amendment, if adopted by the Association, will appear as By-Law 6 of

[1] Annual dues less cost of collection.

Article VII; it is proposed for action at the annual meeting[1] under the provisions of By-Law 1 of Article VIII.

W. D. CAIRNS, *Secretary-Treasurer.*

THE SIXTH SUMMER MEETING OF THE ASSOCIATION.

The sixth summer meeting of the Mathematical Association of America was held by invitation at Wellesley College on Tuesday, Wednesday and Thursday, September 6–8, 1921, in conjunction with, and immediately preceding, the summer meeting of the American Mathematical Society. 146 were present at the meeting, including the following 106 members of the Association:

C. R. ADAMS, Harvard University.
R. C. ARCHIBALD, Brown University.
CLARA L. BACON, Goucher College.
I. A. BARNETT, University of Saskatchewan.
IDA BARNEY, New Haven, Conn.
SUZAN R. BENEDICT, Smith College.
A. A. BENNETT, University of Texas.
E. G. BILL, Dartmouth College.
G. D. BIRKHOFF, Harvard University.
VEVIA BLAIR, Horace Mann School.
G. A. BLISS, University of Chicago.
L. A. BRIGHAM, Boston University.
B. H. BROWN, Harvard University.
R. E. BRUCE, Boston University.
W. D. CAIRNS, Oberlin College.
B. H. CAMP, Wesleyan University.
F. E. CARR, Oberlin College.
A. B. CHACE, Brown University.
J. W. CLAWSON, Ursinus College.
G. M. CONWELL, N. Y. State College for Teachers.
J. L. COOLIDGE, Harvard University.
LENNIE P. COPELAND, Wellesley College.
C. H. CURRIER, Brown University.
C. E. DIMICK, U. S. Coast Guard Academy.
ELEANOR C. DOAK, Mount Holyoke College.
W. P. DURFEE, Hobart College.
J. A. EIESLAND, West Virginia University.
L. P. EISENHART, Princeton University.
H. J. ETTLINGER, University of Texas.
J. A. FOBERG, Penna. Department of Public Instruction.
T. C. FRY, Western Electric Company.
C. A. GARABEDIAN, Harvard University.
H. D. GAYLORD, Browne and Nichols School.
D. C. GILLESPIE, Cornell University.
R. E. GILMAN, Brown University.
O. E. GLENN, University of Pennsylvania.
C. F. GUMMER, Queen's University.
E. R. HEDRICK, University of Missouri.
ARCHIBALD HENDERSON, University of North Carolina.
GOLDIE P. HORTON, University of Texas.
L. A. HOWLAND, Wesleyan University.
L. S. HULBURT, Johns Hopkins University.
E. V. HUNTINGTON, Harvard University.
W. A. HURWITZ, Cornell University.
NELLE L. INGELS, Washington, D. C.
DUNHAM JACKSON, University of Minnesota.
O. D. KELLOGG, Harvard University.
J. P. KELLY, Boston College.
W. D. LAMBERT, Coast and Geodetic Survey.
MARCIA L. LATHAM, Hunter College.
ANNA D. LEWIS, Lake Erie College.
FLORENCE P. LEWIS, Goucher College.
W. R. LONGLEY, Yale University.
A. C. LUNN, University of Chicago.
E. S. MANSON, Ohio State University.
R. M. MATHEWS, Wesleyan University.
NANNIE J. MCKNIGHT, Western High School, Washington, D. C.
T. E. MERGENDAHL, Tufts College.
HELEN A. MERRILL, Wellesley College.
A. L. MILLER, Brookline, Mass.
G. A. MILLER, University of Illinois.
E. B. MODE, Boston University.
C. N. MOORE, University of Cincinnati.
R. K. MORLEY, Worcester Polytechnic Institute.
H. C. M. MORSE, Cornell University.
J. R. MUSSELMAN, Johns Hopkins University.
C. E. NORWOOD, U. S. Naval Academy.
W. F. OSGOOD, Harvard University.
ANNA H. PALMIÉ, Western Reserve University.
ANNA J. PELL, Bryn Mawr College.
L. R. PERKINS, Middlebury College.
E. C. PHILLIPS, Woodstock College.
SUSAN M. RAMBO, Smith College.
W. R. RANSOM, Tufts College.
C. N. REYNOLDS, West Virginia University.

[1] The unavoidable delay in the appearance of the October issue made it impossible to give the necessary notice, in the MONTHLY, one month before the annual meeting; hence it was not possible to adopt this amendment at the annual meeting in December, 1921.

HARRIS RICE, Worcester Polytechnic Institute.
L. H. RICE, Mass. Institute of Technology.
R. G. D. RICHARDSON, Brown University.
E. D. ROE, Jr., Syracuse University.
GEORGE RUTLEDGE, Mass. Institute of Technology.
F. H. SAFFORD, University of Pennsylvania.
HAZEL E. SCHOONMAKER, Gulf Park College.
W. G. SIMON, Western Reserve University.
H. L. SLOBIN, New Hampshire State College.
CLARA E. SMITH, Wellesley College.
D. E. SMITH, Columbia University.
P. F. SMITH, Yale University.
MAY J. SPERRY, Knox College.
J. A. TOBIN, Boston College.
M. O. TRIPP, Wittenberg College.
H. W. TYLER, Mass. Institute of Technology.
ROXANA H. VIVIAN, Wellesley College.
J. L. WALSH, Harvard University.
A. G. WEBSTER, Clark University.
J. K. WHITTEMORE, Yale University.
RUBY WILLIS, Wellesley College.
EUPHEMIA R. WORTHINGTON, Wellesley, Mass.
RUTH G. WOOD, Smith College.
F. S. WOODS, Mass. Institute of Technology.
FRANCES W. WRIGHT, Elmira College.
C. H. YEATON, Oberlin College.
JESSICA M. YOUNG, Washington University.
J. W. YOUNG, Dartmouth College.
MABEL M. YOUNG, Wellesley College.
S. D. ZELDIN, Mass. Institute of Technology.
NOTRE DAME UNIVERSITY, Joseph Donahue, official representative.

Among those who came from more distant parts of the country there were in attendance three from Canada, three from Texas, two from Missouri and one each from Minnesota, Mississippi, and North Carolina.

The committee on local arrangements planned most effectively for the comfort of guests and this was recognized in a motion presented by Professor C. N. Moore and heartily adopted in the Thursday morning session. Well-appointed quarters were provided in Tower Court, one of the magnificent dormitories, the members using this building also for their meals, for committee meetings and for frequent social gatherings. On Tuesday afternoon an opportunity was afforded to visit the private gardens of the beautiful Hunnewell estate; in the evening an organ recital was given by Professor Macdougall of Wellesley College, and parties were afterward taken on a visit to Whitin Observatory. On Thursday afternoon a visit was made to the unusual collection of butterflies belonging to the Denton brothers in Wellesley. After the close of the Society program Friday noon, automobiles were placed at the disposal of all who wished to remain, for a ride to the historic spots in Lexington and Concord.

The joint banquet of the Society and the Association was attended by about 120 members and visitors. President Ellen F. Pendleton gave a gracious welcome to Wellesley's guests, and under the toastmastership of President Bliss of the Society speeches were made by President Miller of the Association, Professors Huntington, Jackson, Hedrick, Lewis, Eisenhart and Webster.

GIFT TO THE ASSOCIATION FOR THE PUBLICATION OF MATHEMATICAL MONOGRAPHS.

The outstanding feature of the Wellesley meeting was the announcement of a notable gift made to the Association. The nature and the terms of the gift cannot be presented to our members and to the mathematical fraternity more clearly than in the admirable letter of transmission from the donor:

LA SALLE, ILL., August 24, 1921.

PROFESSOR H. E. SLAUGHT,
Chicago, Illinois.

Dear Professor Slaught:

Confirming our recent conferences regarding the publication of a series of monographs on mathematical subjects, to be prepared and published by The Mathematical Association of America, I as trustee for the Edward C. Hegeler Trust Fund hereby offer a yearly contribution to the Association of twelve hundred dollars ($1200) for a period of five years payable on January first of each year and beginning in January 1922, these funds to be under the complete control of The Mathematical Association of America. It is understood that the distribution of these monographs to the members of the Association will naturally be conducted through its own channels, but for their distribution to the general public the Open Court Publishing Company gladly offers its services.

If at the end of five years this project shall have proved successful it is my intention to then give to the Association a permanent endowment fund, and I will so direct my legal representatives, which will yield at least twelve hundred dollars annually, hoping that this will be sufficient income for the publication of two or three monographs a year, as long as this form of publication appears desirable and best suited to the diffusion of mathematical and formal thought as contributory to exact knowledge and clear thinking, not only for mathematicians and teachers of mathematics but also for other scientists and the public at large. But nothing is hereby intended to restrain the Association from the use of this income for the promotion of other mathematical publications if at some future time it should be deemed best to cease the publication of such monographs.

The Open Court Publishing Company has also been working for the diffusion of mathematical knowledge through a related channel, namely, by making certain mathematical classics available to the public through reprints, and in the furtherance of this ideal it is hoped that its coöperation with the Association will prove desirable and mutually helpful, but it is not intended to bind either organization.

It is my hope that The Mathematical Association of America will be successful in editing such a series and that it will set a standard of simplicity and clearness not before attained by mathematical writers in English.

Trusting that this donation may be of general benefit, I am

Yours very truly,
(Signed) MARY HEGELER CARUS,
Trustee Edward C. Hegeler Trust Fund.

Appropriate action has been taken by the Trustees in formally accepting this gift and expressing their grateful recognition of this notable opportunity for an advance in American mathematics. Further, by a motion adopted unanimously in the morning session of the Association, Professor Smith and the Secretary were instructed to send a letter to Mrs. Carus embodying the Association's keen appreciation of her generous gift. This letter follows.

Dear Mrs. Carus:

At a meeting of the Mathematical Association of America, held at Wellesley College September 7, 1921, there was read to the members your letter of August 24, 1921, announcing the notable gift made by you for the purpose of enabling the Association to publish a series of mathematical monographs. Although it was further announced that the Trustees had formally accepted the trust, together with the obligations attendant thereon, and that they had expressed to you their appreciation of your generosity, the Association, desiring also to express its own sentiments upon this memorable occasion, passed the following resolution and delegated the undersigned to convey the same by letter to you:

Resolved that the Mathematical Association of America, having been advised of the munificent gift of Mrs. MARY HEGELER CARUS as Trustee of the Edward C. Hegeler Trust Fund, desires to express its sincere appreciation of her action and to assure her of its determination to make every effort to carry out the trust in accord with the desires which she has so clearly set forth. The Association joins with her in the expression of the belief that it is both possible and feasible to

present to a large body of intelligent readers a clear and satisfying view of the leading branches of modern mathematics which, under present conditions, they are unable to obtain. To this end the Association pledges its efforts to secure an efficient board of editors who will carry out the project, mindful of its responsibilities and determined to make of the series the success for which we all confidently hope.

Yours very truly,

DAVID EUGENE SMITH,
W. D. CAIRNS.

President Miller presided at the sessions of the Association, save that Vice-President Archibald replaced him for a part of the Wednesday morning session. Presidents Bliss and Miller occupied the chair at the joint sessions of Wednesday afternoon and Thursday morning respectively. The program was formulated under the chairmanship of Professor W. R. Longley and the committee's able rendering of its duties was commended in Professor Moore's motion. When it was announced at the final session that Professor A. A. Bennett had been appointed Editor-in-Chief of the AMERICAN MATHEMATICAL MONTHLY to succeed Professor Archibald, whose resignation had been accepted by the Trustees to take effect at the close of the present volume, the following resolution was offered by Professor Jackson and heartily adopted by the Association:

"Voted, That the Association place on record its appreciation of the faithful, enthusiastic, and scholarly services of Professor Archibald as Editor-in-Chief of the MONTHLY, constituting a contribution to its history which no other man could have made and a significant stage in its advancement toward a permanent place among the most important mathematical journals."

The following papers were read, the last two being given in the joint session of the Society and the Association.

(1) "Freshman Mathematics" by Professor W. F. OSGOOD, Harvard University.

General discussion, led by Dean E. G. BILL, Dartmouth College, and Professor CLARA L. BACON, Goucher College.

(2) Report of progress of the National Committee on Mathematical Requirements, by Professor J. W. YOUNG, Dartmouth College.

(3) "Religio mathematici"—retiring presidential address by Professor D. E. SMITH, Teachers College, Columbia University.

(4) "Synthetic projective methods of generating cubic and quartic curves" by Professor HELEN A. MERRILL, Wellesley College.

(5) "A few questionable points in the history of mathematics" by Professor G. A. MILLER, University of Illinois.

(6) "A simple form of Duhamel's theorem and some new applications" by Professor H. J. ETTLINGER, University of Texas.

(7) "Some mathematical aspects of the theory of relativity" by Professor JAMES PIERPONT, Yale University.

(8) "The place of the Einstein theory in theoretical physics" by Professor A. C. LUNN, University of Chicago.

Abstracts of the papers and discussions follow below, the numbers corresponding to the numbers in the list of titles.

1. Calculus is now rather generally accepted as an essential part of the curriculum of the freshman year, the more readily for the fact that the influential class of technical schools require trigonometry and solid geometry for entrance. A generation ago a leisurely treatment of calculus was given in the sophomore year, largely formal as represented in the texts of that period, the applications being for the most part geometrical and very closely following the prototype. Next followed what may be called the Perry movement. Professor Osgood feels it imperative to counsel a middle course. Numerous important principles were presented in his mapping out of the freshman course. The applications of the differential calculus to maxima and minima and allied topics should be included in a first course, and here as in later courses mathematicians must lay hold of the definite integral as the limit of a sum as an illuminating feature of the calculus. Formal work is to be carried to the point that for the easier applications the student need not be held up by the necessary differentiation or integration.

As to the amount of work that can be included, the speaker noted that in S. A. T. C. days he found that in one third of a year, in about thirty periods, it was possible to take up the vital parts of analytic geometry, including some applications, for example, the geometrical determination of points on an ellipse, etc. It was not a course for *seeing* analytic geometry; nevertheless it transformed the freshman from a high school boy into a college student. Then it was possible to begin the calculus.

Early in this course it is necessary for the pupil to see the derivative at work; the question of what the derivative is can be gained by seeing what it is for. The place for introducing the infinitesimal depends on the individual, Professor Osgood finding it convenient to put it at the end of the first group of differentiations before transcendental functions, this with full recognition of the fact that college teachers should not feel forced to turn out a finished product in their classes. For example, briefly stated, we may begin with the familiar definition $\lim_{\Delta x \to 0} \Delta y/\Delta x = D_x y$, and since $\Delta y/\Delta x = D_x y + \epsilon$, ϵ being a correction term, we have as a consequence $\Delta y = D_x y \cdot \Delta x + \epsilon \Delta x$, whence we get $dy = D_x y \cdot \Delta x$. Then we prove that $dy = D_x y \cdot dx$, etc.

Next comes the differentiation of trigonometric functions, with the addition of which there is ample material for drill in the formal side of differentiation. We may now introduce additional problems in maxima and minima involving trigonometric functions; here is a natural place to treat the analytical part of trigonometry.

The student should now be given the great advantage which the calculus affords in curve tracing. The differentiation of logarithmic functions comes next. Inverse functions need not enter until the work on integration.

But here the new mathematics diverges sharply from the old. We must now get over the subject of indefinite integrals, treating the necessary definitions and

formulas, for example, the clear realization of the fact that these equations all have the same meaning:

$$U = \int u dx; \qquad D_x U = u; \qquad dU = u dx.$$

The several elementary principles of differentiation (derivative of a sum, etc.) must needs be given in this course as much as in the older course. The number of formulas can however be limited to less than twelve, the others being deduced by several methods, the most important of which are *gumption* and substitution.

Professor Osgood then introduces simple problems in physics, explaining the method of defining elementary work for a variable force, in extension of the pupil's high school principle of work, and thus leading up to the use of Duhamel's theorem. He closed his address by saying that there never has been a time when teachers were more at a premium than today; that teachers need continually to adapt their methods to the various conditions, for example, that women students are not as a class vitally interested in the problems of physics; that there should be inculcated in our pupils as fully as possible: (1) a broad and deep interest in mathematical analysis for its own sake, (2) a sympathetic attitude toward the outstanding problems of physics, (3) judgment on the scientific side of choosing that which is genuine and fundamental and of avoiding that which is impossible of attainment, and also, judgment as to the proper presentation of this material.

Dean Bill stated that the calculus is best adapted to present those methods that are fundamental. The pupil should (1) acquire the ability to analyze a problem, (2) have proper tools, (3) seek the one prime requisite of accuracy, the average teacher being apt to pass a student on the ground that he has the right idea, even though he be woefully deficient in expressing this idea accurately. To accomplish this, (1) we must start the student in a correct manner from the outset, it being a psychological error to begin with mathematical reviews rather than to stimulate the student by taking up topics that are at least ostensibly new; (2) we must recognize that the pupil, like ourselves, cannot see many steps in advance and that it is by far better to master one subject fully than to cover many subjects and so to contribute to the great American trait of superficiality. It may be on this last ground that one may criticize Professor Osgood's course for trying to cover too much in the first short calculus course; it may be possible however to cut down the material in trigonometry and in conic sections sufficiently to gain time for the calculus.

Miss Bacon remarked that Professor Osgood's course as outlined was stimulating for those who have presented advanced entrance credits or are specially gifted in mathematics, but that it is difficult to do justice to trigonometry and analytic geometry and to bring all of this material into the second semester, particularly in a three hour course. She advocated having two freshman courses, one similar to that suggested by Professor Osgood, consisting of analytic geometry in the first semester and calculus in the second, the other for the ordinary freshman, less mature mathematically, who offers only plane trigonometry and elementary algebra, the course to consist of trigonometry for the first semester and analytic geometry for the second, leaving the calculus for the sophomore year.

Professor Longley spoke of the great variation in the notion of freshman mathematics, as indicated by the results of a questionnaire sent to about thirty colleges and universities from Yale; for example, the amount of time devoted to advanced algebra varied from zero to fifty-five hours, trigonometry varied greatly, conics varied from zero to twenty-four lessons. Only a few of these taught as much as a half year of freshman calculus. The reports differed so much from each other that no general conclusions could be derived from the data.

Professor Glenn expressed his pleasure at hearing of freshman teaching based on genuine theory rather than pedagogical convenience and observed that something may still perhaps be said as to giving a freshman more in some parts of his first course than he can stand.

In reference to students who had not had trigonometry, Professor Osgood said that while the numerical portion of trigonometry should not be curtailed, the analytical portion can almost all be treated in connection with calculus; that, for example, trigonometry might be taught for five weeks, such topics as the addition theorem being touched upon in the first treatment, the remainder of the first half year reserved for analytic geometry, thus using the second half year for calculus with a fuller treatment of some topics of the first half year.

Professor Ransom, in commenting on Professor Longley's remarks, felt sure that the general tendency of colleges over the country is toward two types: one the old-fashioned type consisting of extracts from college algebra, solid geometry and analytic geometry, the other, now emerging, including parts of trigonometry, analytic geometry and calculus.

2. The National Committee held its last meeting under its present form of organization on September fifth. One phase, at least, of its work has come to an end. In view of several previous reports made to the Association, it hardly seems necessary at this time to review in detail the work of the Committee. The present report may well be confined to a statement regarding present conditions and plans for the immediate future. The manuscript of a summary of the final report of the Committee has been sent to the U. S. Bureau of Education for publication. This summary, which will constitute a bulletin of some eighty pages, virtually presents the first part of the complete report. It contains the following chapters:

I. A Brief Survey of the Report
II. Aims of Mathematical Instruction—General Principles
III. Mathematics for Years Seven, Eight and Nine
IV. Mathematics for Years Ten, Eleven and Twelve
V. College Entrance Requirements in Mathematics
VI. List of Propositions in Plane and Solid Geometry
VII. The Function Concept in Elementary Mathematics
VIII. Terms and Symbols in Elementary Mathematics,

and also a brief synopsis of the remaining chapters of the complete report. It is expected that this summary will appear late in November or early in December.

It was the original intention of the Committee to publish its complete report

also through the U. S. Bureau of Education. It was found, however, that this would involve a delay of two or three years in view of the fact that it would have been necessary for the Bureau of Education to issue the report in parts extending over a considerable period of time. It is hoped at present that sufficient funds will be obtainable to publish the report during the winter and to distribute it free of charge to all who are sufficiently interested to ask for it. The complete report will constitute a volume of about five thousand pages. In addition to the chapters listed in the summary, it will contain an account of a number of investigations instituted by the Committee. Among these may be mentioned:

The Present Status of Disciplinary Values in Education
A Critical Study of the Correlation Method Applied to Grades
Mathematical Curricula in Foreign Countries
Mathematics in Experimental Schools
The Use of Mental Tests in the Teaching of Mathematics
The Training of Teachers of Mathematics.

There will also be included an extensive bibliography on the teaching of mathematics.

In closing this phase of its work, the Committee desires to extend its most cordial thanks to all the individuals and organizations that have helped. The response secured by the Committee to its appeal for assistance in solving the many problems facing it has been extremely enthusiastic and gratifying. This leads the Committee to look forward to the future optimistically. The real work for which the Committee was appointed may be said to begin with the publication of its report rather than to end with it. Continued enthusiastic activity on the part of all individuals and organizations concerned with the teaching of mathematics is needed over a period of many years to put the recommendations of the Committee into effect, to test their validity and to modify them in ways that experience shows to be desirable. In order to be of assistance in this direction, the Committee hopes to be able to maintain an office with a certain amount of clerical help during the next few years so that it may continue to act as a clearing house for ideas and to stimulate the discussion of problems relating to the teaching of mathematics among the nearly one hundred organizations that have in the past been actively coöperating with the Committee. Members of this Association are strongly urged to continue their support and activity in connection with the cause for which the National Committee has been laboring.

Following Professor Young's report the session adopted by a hearty vote a motion of Professor Coolidge recognizing the great importance of the series of reports, their potential influence for good in the cause of mathematics if these reports come fully to the attention of American teachers, and commending the committee members and those collaborating with them for the high order of ability and thoroughness shown in these reports.

3. President Smith's able retiring address is published elsewhere in this issue of the MONTHLY.

4. The methods ordinarily given in the texts for the generation of cubics by two flat pencils of the first and second order respectively can be greatly shortened by substituting for one or both of these pencils a flat pencil of the first order in involution. Schröter has proved that the general cubic may be generated by two such pencils related projectively and having a common corresponding ray, while, if the common ray is not corresponding, a quartic is generated. Unicursal cubics may be generated by relating projectively two flat pencils of the first order, one of which is in involution, and then special cubic curves may be obtained very readily from systems of coaxial circles.

Miss Merrill's paper, embodying a method which has proved especially interesting even to first year students in two or three colleges, and which gives points on the curves very expeditiously, since these ordinarily are given in sets of four, may be expected in an early number of the MONTHLY.

5. Professor Miller's paper, in effect, appeared in *Scientific Monthly*, September, 1921.

6. Professor Ettlinger presented a very simple form of Duhamel's theorem by means of the following geometric lemma: Let the interval I: $a \leqq x \leqq b$ be divided into n equal subdivisions, I_{in}, of length, $\Delta x_n = (b - a)/n$, $(i = 1, 2, \cdots, n)$. Upon I_{in} as base, draw a rectangle of area, R_{in}, and height h_{in}, such that $|h_{in}| \leqq M$ for all values of n. If $\overline{P}$ is any fixed point of I, there is for each value of n, at least, one rectangle whose base contains $\overline{P}$. If for each fixed $\overline{P}$, the altitude of this rectangle, h_P, approaches zero as $n \to \infty$, then $\lim_{n\to\infty} \sum_{i=1}^{n} R_{in} = 0$.

The advantages of this form are fourfold: (1) it is simple enough for a first course in the calculus, (2) it avoids the question of double limits and uniform convergence, (3) it is more general than earlier contributions, (4) it is easily applied.

A simple proof of the integrability of a continuous function is given by means of this formulation. Finally, new applications are made to the transformation of a double integral and to the solution of an integral equation.

The discussion of this paper turned rather upon the wisdom of using Duhamel's theorem or a substitute; it was observed however that Professor Ettlinger apparently avoided the use of uniform convergence, but that logically this principle must enter into any proof of Duhamel's theorem or its equivalent.

In his inimitable manner Professor Huntington laid, as he said, a wreath on the tomb of Duhamel's theorem for the very good reason that he believed it to be dead, had great respect for it and had a belief in its immortality. He illustrated how one can set up a definite integral with perfect rigor without the use of Duhamel's theorem, his full presentation having been given in this MONTHLY, *1917*, 271–275.

Professor Bliss discussed the problem of the total attraction due to a thin rod, using the substitute for Duhamel's theorem which he published in the *Annals of Mathematics* for December, 1914.

Professor Birkhoff mentioned his increasing attachment to Duhamel's theorem. He has found such an illustration as this illuminating and easily understood: If I am calculating the cost of building from the estimates of various sub-contractors, each being correct to within 10 per cent., I am certain that my whole estimate is correct to within 10 per cent. He considers it wise to use, not the more difficult problem of attraction, but a simpler application, the length of an arc. After simple examples he tells his students that it is well for them to read Duhamel's theorem, his experience being that a gratifying number of students will work their problems by actually using Duhamel's theorem.

7. Professor Pierpont's purpose, achieved in a wonderfully clear manner, was to present in simple form the main mathematical facts on which Einstein's theory rests. As an extension of two- and three-dimensional concepts he described an n-way space or manifold, a hypersurface, distance and its connection with the theory of bilinear forms, angle, volume, geodetic curve, and the term "curvature of space," which is the generalization of the Gaussian curvature of a surface with its invariant $1/(R_1R_2)$. After he explained the mathematical notion involved in tensors and their properties, he showed how by generalizing the vector treatment of Maxwell's equations and similar equations Einstein derived his fundamental equations for a universe in which the geometry of a space is determined not merely through the conventions as to length and time, but by all the material that exists in this space. He finally sketched the still more generalized theory of Weyl, wherein any existing amounts of electricity and magnetism serve also to determine the geometry.

It will be of great service to all interested in the Einstein theory if, as is expected, Professor Pierpont's very helpful introduction to the difficulties of the mathematics of relativity may be made available through an American journal.

8. Because of the nature of the subject it is difficult to outline Professor Lunn's presentation of the place of the Einstein theory in theoretical physics. He sought in the main to give the physical interpretation of the mathematical notions and principles involved in Einstein's theory, indicating for example how the notion of n dimensions may be realized experimentally. A considerable part of the address had to do with a description of the vector notation and expressions used in the development of the theory.

Meeting of the Board of Trustees of the Association.

Ten members of the Board were present at this meeting.

(*a*) The Trustees received the formal letter from Mrs. Mary Hegeler Carus already presented in this report. The Board voted formally to accept the gift and to instruct the Secretary to send to Mrs. Carus the acceptance and the expression of gratefulness with which the Trustees regard this gift. It was voted to express the obligation of the Board to Professor Slaught for his large share in inspiring this gift and for his continued services to the Association in

this great enterprise. A committee, consisting of Professors Slaught, Veblen and Cairns, was appointed to nominate to the Trustees an editorial board on monographs and to formulate a statement of the powers of this board.

(*b*) In view of the resignation of Professor Archibald as Editor-in-Chief of the MONTHLY, to take effect with the completion of the present volume for 1921, Professor A. A. BENNETT of the University of Texas was appointed as his successor.

(*c*) The following seventy persons and five institutions, on applications duly certified, were elected to membership:

To individual membership.

L. C. BAGBY, A.B. (Washburn). Head of dept. of math., Ottawa Univ., Ottawa, Kans.
A. J. BARRETT, A.B. (Arkansas). Instr., U. S. Naval Acad., Annapolis, Md.
E. T. BROWNE, A.M. (Virginia). Asst. prof., Trinity Coll., Hartford, Conn.
JULIA A. BURRELL, A.B. (Rice Inst.). Galveston, Tex.
W. R. BURROWS, Medical student, Univ. of Chicago. Halstead, Kans.
MILDRED E. CARLEN, Student, Brown Univ., Secretary to the Editor-in-Chief of the MONTHLY, Providence, R. I.
E. L. CARR, A.B. (Ewing Coll.). Prof., Shurtleff Coll., Alton, Ill.
MARGARET F. CHAPMAN, A.B. (Wisconsin). Instr., State Normal School, Milwaukee, Wis.
E. F. CHURCH, C.E. (Syracuse). Prof., Penna. Military Coll., Chester, Pa.
LENA R. COLE, A.M. (Missouri). Head of dept. of math., Central Coll., Lexington, Mo.
ALBERTUS DARNELL, Ph.B. (Michigan). Head of dept. of math., Detroit Junior Coll., Detroit, Mich.
H. A. DAVIS, Asst., W. Va. Univ., Morgantown, W. Va.
A. K. DENNY, M.S. (Lincoln). Prof. and registrar, Lincoln Coll., Lincoln, Ill.
L. E. DONNELLY, Student, Univ. of Chicago, Chicago, Ill.
B. F. DOSTAL, A.M. (Indiana). Asst. prof., Univ. of Denver, Denver, Col.
FRANC C. EARHART, B.S. (Lenox). Dean of Women, Lenox Coll., Hopkinton, Iowa.
LOUIS FELDMAN, B.S. in E.E. (Cooper Union). Brooklyn, N. Y.
J. C. FIELDS, Ph.D. (Johns Hopkins). Prof., Univ. of Toronto, Toronto, Canada.
GLADYS H. FREEMAN, A.M. (Chicago). Instr., State Coll. of Washington, Pullman, Wash.
T. C. FRY, Ph.D. (Wisconsin). Mathematician, Research Labs., Amer. Tel. and Tel. Co., and Western Electric Co., New York, N. Y.
LESLIE J. GAYLORD, A.B. (Lake Erie). Instr., Agnes Scott Coll., Decatur, Ga.
GLADYS E. C. GIBBENS, Ph.D. (Chicago). Instr., Univ. of Minn., Minneapolis, Minn.
G. W. GORRELL, A.M. (Ohio State). Prof., Colo. School of Mines, Golden, Colo.
R. F. GRAESSER, A.B. (Illinois). Asst., Univ. of Illinois, Urbana, Ill.
LILIAN HACKNEY, A.B. (W. Va. Univ.). Head of dept. of math., Marshall Coll., Huntington, W. Va.
W. R. HALE, A.M. (Alabama). Asst. prof., Colo. School of Mines, Golden, Colo.
CLARA L. HANCOCK, A.M. (Iowa). Teacher, Junior Coll., Virginia, Minn.
D. C. HARKINS, A.B. (W. Va. Univ.). Asst., W. Va. Univ., Morgantown, W. Va.
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To institutional membership.

UNIVERSITY OF NOTRE DAME, Notre Dame, Ind.
ANTIOCH COLLEGE, Yellow Springs, Ohio.
SOUTHEASTERN STATE NORMAL SCHOOL, Durant, Okla.
VIRGINIA POLYTECHNIC INSTITUTE, Blacksburg, Va.
COLLEGE OF WILLIAM AND MARY, Williamsburg, Va.

(*d*) The President was empowered to appoint a representative of the Association on the Council of the American Association for the Advancement of Science, in addition to the Secretary-Treasurer, who is already a member; President Miller has appointed to this position Professor H. L. RIETZ of the University of Iowa.

(*e*) Professors A. A. BENNETT, H. P. MANNING and H. E. SLAUGHT were appointed as the Committee on Publications beginning with the volume of the MONTHLY for 1922.

(*f*) A committee consisting of three Association members of the National Committee on Mathematical Requirements, Professors J. W. YOUNG, C. N.

MOORE and H. W. TYLER, was appointed for the purpose of coöperating with a similar committee, from the Society for the Promotion of Engineering Education, in an investigation of the teaching of mathematics in engineering schools.

(*g*) The Association received an invitation from the department of mathematics of Cornell University to hold its summer meeting at Ithaca in either of the years 1923 or 1924, in conjunction with the American Mathematical Society.

W. D. CAIRNS, *Secretary-Treasurer.*

MATHEMATICAL PROBLEMS IN THE WORK OF THE UNITED STATES COAST AND GEODETIC SURVEY.[1]

By WALTER D. LAMBERT and OSCAR S. ADAMS,

Division of Geodesy, U. S. Coast and Geodetic Survey.

In the office of the United States Coast and Geodetic Survey there are three divisions that employ mathematics to a considerable extent in the routine work which they perform. These are the Division of Geodesy, the Division of Tides and Currents, and the Division of Terrestrial Magnetism. We shall aim to give some account of the nature and extent of the mathematics required as a tool in each of these three divisions.

The work of the Division of Geodesy is almost entirely mathematical and the problems that are presented for consideration are many and varied. The routine work consists of the computation and adjustment of triangulation, the computation of astronomical observations, the computation of gravity observations, and the computation and adjustment of observations in precise leveling. This is a bill of fare that should be sufficiently comprehensive to suit the taste of any one who is mathematically inclined (or should we say, in accordance with popular belief, mathematically deranged?)

The computation of triangulation and of astronomical observations requires a thorough training in trigonometry with especial attention to the use of logarithmic tables. This training should be directed toward the attainment of both speed and accuracy, with a considerable premium placed on accuracy. It may be thought that common sense would suffice for the use of tables, and so it would seem; but, if so, good common sense is oftentimes sadly lacking in men when they graduate from college. In the examinations given by the Coast and Geodetic Survey, mistakes are made by candidates that are so utterly absurd that one can hardly explain the psychology of the errors.

The adjustment of observations by the method of least squares forms the greater part of the purely mathematical work in the general routine of the geodetic division. This adjustment finds its most extensive field of application when applied to triangulation. The regular method employed in the Survey for this purpose is that of adjustment with condition equations as distinguished from

[1] Read at the meeting of the Ohio Section of the Mathematical Association of America, March 25, 1921.

observation equations. The development of such condition equations for closures in length, azimuth, latitude, and longitude, requires considerable skill in the application of analysis to practical problems. These equations must be rigid enough to fulfil their purpose, but at the same time they must be as simple as possible so as to obviate the performance of any undue labor in the practical formation of the equations for any specific piece of triangulation.

The use of observation equations in the adjustment of triangulation, tried by other countries, has been found to have certain advantages. This Survey has worked out the necessary formulas and published them with examples for their use, but has not so far applied the method extensively to its own work. For the adjustment of differences of elevation, however, whether based on spirit leveling or trigonometric leveling, it is the practice of the Survey to use the method of observation equations, though it is quite practicable to use the method of condition equations. In the treatment of gravity observations and astronomical observations, observation equations are more frequently used than equations of condition.

The two largest adjustments that have ever been made in the Survey consisted of 264 and 284 equations. The amount of labor involved in the formation and solution of such a set of equations is enormous. In addition, one small error may require a month or two of hard work to make the necessary correction. It has been proposed to include all of the primary triangulation in the United States in one piece for scientific purposes, but since this would involve an adjustment consisting of between three and four thousand equations it has not as yet been attempted.

Whatever may be our opinion of the theory of least squares as founded upon probability, it must be recognized that it gives an exceedingly simple method for reconciling discrepancies in observations which, though small and practically insignificant, are yet troublesome because of the inconsistencies that they introduce. When one considers carefully, it is astonishing what degree of accuracy can be attained by careful work. The requirement for precise triangulation is that the average closing error of the triangles should be one second or less. Now on the largest circle used in horizontal angle measures (the 12 inch circle) one second represents about 58 one-millionths of an inch or about one and one half microns. With this fact in mind it is truly remarkable that such accuracy can be attained. By the adjustment these small outstanding discrepancies are reconciled and all values which have a check give one and the same result however computed.

Besides this regular routine work, the geodetic division prepares special publications on the theory of cartography, on triangulation, on the theory of gravity and isostatic compensation, and on various other phases of geodesy and geophysics. In this work practical applications may be found for many branches of mathematics. The whole subject of cartography comes finally under the head of the differential geometry of curves and surfaces. The subject of conformal mapping is founded directly upon the theory of the functions of a complex

variable and forms the field for one of the most interesting practical applications of this branch of pure mathematics. The attempt to treat any of the projections in this class without consideration of its foundation upon the general theory of functions of a complex variable leads always to lack of clearness and definiteness of conception. Lagrange and Gauss have contributed much towards the generalization of the treatment of this kind of mapping. The equal-area mapping is a special case of equal-area representation of one surface upon another such as is treated in extended works on differential geometry. The perspective projections are practical applications of the principles of projective geometry. It is needless to say that trigonometry and calculus, both differential and integral, are employed in all of this cartographic work. Developments in Taylor series and especially those in Fourier series are frequently employed.

When tables are computed for any one of the projections, extensive logarithmic trigonometric computations have to be performed. In this work the judgment must be continually on the alert to determine the necessary exactness of the calculations required to attain the end desired. During the period of the war two extensive tables of this kind were prepared in the Division of Geodesy; the results are given in *Lambert Projection—Tables for the United States* (U. S. Coast and Geodetic Survey, Special Publication no. 52) and *Grid System for Progressive Maps in the United States* (U. S. Coast and Geodetic Survey, Special Publication no. 59). Both of these tables were prepared at the request of the army authorities and they were intended primarily for military purposes.

The tidal problem is still very far from a satisfactory solution despite the efforts of many of the ablest mathematicians since Newton's time. For purposes of prediction a modified form of the statical theory is used. This requires something of spherical harmonics and a great deal of development in series very similar to the developments needed in the lunar and planetary theories of dynamical astronomy, though not nearly so extensive. In questions of hydrodynamics involved in the tidal theory as so far developed, use is made (among other branches) of spherical harmonics, Fourier series, functions of a complex variable, Bessel functions, partial differential equations, and functions too recently introduced or too special to have any well-known name. Even the theory of numbers has been applied indirectly to tidal problems, for some of its theorems were found useful in deciding on the number of teeth to put on the gears of a tide-predicting machine.

The routine of computations regularly made in the Division of Tides and Currents calls for frequent use of the trigonometric functions in all four quadrants. Perhaps this is as good a place as any to call to mind what most teachers of trigonometry will sadly admit to be true, namely, that many students end their formal study of that subject with the idea that the signs of the trigonometric functions in the various quadrants and the process of looking up functions of an angle greater than 90° are great mysteries, so mysterious that even the rules for these processes are difficult to memorize.

In the Division of Terrestrial Magnetism the mathematician must be thor-

oughly familiar with his trigonometry and use of logarithms and must be prepared to use the principles of calculus and Fourier analysis in the reduction and discussion of the results. As the quantities measured are subject to continuous fluctuations, the results cannot be subjected to rigid adjustment as in the case of triangulation, but the method of least squares can frequently be used to advantage for interpolation or for separating errors of observation from fluctuations of which the character is established. Some of the fluctuations of the earth's magnetism are of a periodic character and the harmonic analysis may be used to deduce from the observations the periods and amplitudes of fluctuations which may be correlated with other terrestrial or cosmical phenomena. In the graphical representation of the distribution of the earth's magnetism over a given area it is sometimes desirable to determine the regular distribution corresponding most nearly to the observed results, and this may be done by the methods of least squares, using an analytical expression involving the first and second powers of the latitude and longitude.

While much of the work of the Division of Terrestrial Magnetism is of a routine character, involving little more than the application of arithmetic and logarithms to the use of formulas, yet even for this work the computer must understand the derivation of the formulas, know how to modify them when the character of the data justifies the use of approximations, and regulate the refinement of computation by the degrees of accuracy to be expected.

In the design of instruments for the Survey, various questions of mechanics and mathematical physics arise.

If then we consider not only the daily routine but also the theory underlying that routine and the necessity for improving methods and advancing knowledge, we shall not be surprised at the number and variety of the mathematical theories that are laid under contribution. But little more than an imperfect catalogue can be given here. Time and space would fail for any detailed treatment.

In Geodesy proper we have solid analytic geometry, differential geometry and elliptic functions in connection with measurement of the terrestrial spheroid. In connection with gravity problems, we need spherical harmonics, including zonal harmonics of the second kind, and in some cases curvilinear coördinates and ellipsoidal harmonics. In some problems integral equations are involved. In questions of geophysics intimately related to geodesy proper, such as the transmission of earthquake waves, integral equations are again involved. The transmission of earthquake waves requires, of course, the mathematical theory of elasticity as does also the mechanical problem of the variation of latitude. In questions of mapping, as has already been mentioned, differential geometry and the theory of functions of a complex variable and of trigonometric series are needed.

No matter how difficult or how advanced the theory on which a formula is based, the numerical application of that formula usually comes down to the use of tables and the four rules of arithmetic. In certain cases where arithmetical work on a large scale is to be done, it is possible to utilize the services of people

who apply the rules of arithmetic to the data with little understanding of the reasons for what they are doing. Those who can do only this kind of unskilled labor—for it is mental labor corresponding to pick-and-shovel work in the field of manual labor—have been found to be unprofitable in the Coast and Geodetic Survey. They require too much supervision while at work and the supply of work that they can do is not continuous enough to ensure steady employment for them. For while there is much routine numerical work in the Coast and Geodetic Survey, there is a rather surprising amount of work that cannot be done by any cut-and-dried method. Peculiar conditions frequently arise in field work that must have special treatment in the reductions, and requests are continually coming in for information that the Survey would naturally supply, but which is not ground out by any of the regular machinery. Moreover, although a certain stability in form and method is desirable where great masses of data are handled, the results being published serially as they become available, nevertheless, no organization can keep on doing the same old thing in the same old way without suffering retrogression and decay. The Survey therefore seeks to learn and assimilate improved methods developed by other organizations doing similar work, to devise new methods on its own account and to advance knowledge in its own field.

Perhaps you will wonder, after hearing the foregoing list, not what branches of mathematics the Coast and Geodetic Survey needs, but what branches it does not need. You will find a few, however, that have not been mentioned. Perhaps the reason that some have been left out is that the right person has not come along to apply them; it may appear in the future that some branches of mathematics whose applicability to problems of mathematical physics is little known or not even suspected will turn out to be just what is needed to solve some problem in the work of the Coast and Geodetic Survey.

From their experience in the bureau, it seems to the authors desirable to lay stress on the following topics in elementary mathematics. These topics are important not alone in the work of the Coast and Geodetic Survey, but in other applications of mathematics to scientific and technical problems, and are not without cultural and disciplinary value. These concluding paragraphs are not to be taken as criticisms of present-day curricula and methods of teaching. The authors have both had some experience in teaching, and realize that, especially in pedagogical matters, it is easy to give advice, but much less easy to put it in practice.

1. *Numerical* calculation, first, with reference to *accuracy*, and second, *to economy of effort.* Very few young men just out of college or technical school have much idea of proportioning the accuracy of one part of a calculation to that of the other parts. They will cheerfully add together two quantities, one known to many decimal places, and the other known to much fewer, or they will multiply together two quantities, each known say to only three significant figures, and then retain the sixth significant figure of the product as if it meant something. It has never occurred to them that when the difference of two nearly equal quantities

is taken the number of accurate significant figures left is much reduced. These things are all simple enough, and are readily seen to be true once attention is drawn to them, but to many they seem to be considerations of a wholly new sort.

2. *Analytical trigonometry.* Somehow the signs of the trigonometric functions in the various quadrants should be taught so as to be retained, also the process of looking up functions of an angle greater than 90°, and the fact that the angle corresponding to a given function is not unique. Furthermore, a working knowledge of various trigonometric transformations is useful in applied mathematics.

3. *Approximations.* It is desirable to give students who are to apply their mathematics some instruction and training in the use of approximate formulas, in order that they may not distrust an admittedly approximate formula in those cases where it legitimately applies, and in order that, on the other hand, they may realize the limitations of such a formula and not try to make it do more than it is able to do.

AMONG MY AUTOGRAPHS.

By DAVID EUGENE SMITH, Columbia University.

12. The Marquise du Chastellet.

One of the letters in my collection is of special interest because of two lines. The letter is as follows:

> We have arrived in good health, Monsieur, but with disquietude which your own health causes us. I hoped that M. Cléraut would have had news from you. I beg of you to give me information about yourself, and to tell me if we shall see you soon. You do not realize the interest that I have. M. de Voltaire charges me to present to you his most affectionate compliments; and as for me, I am, with very profound friendship, Monsieur, your very humble and very obedient servant
>
> Breteuil du Chastellet.
>
> Paris, August 31, 1744.
>
> I hope that you will not forget our little affair.

So Voltaire sends by the Marquise du Chastellet (to adopt her own spelling of the name) his affectionate compliments to some one whom we do not know. She was entirely frank about delivering the message. Voltaire, who six years before this had essayed to make Newton intelligible to the French people, was with her. In 1735 they had begun their work together at the Chastellet country place at Cirey,—"that pitiless bore, the unfortunate Marquis," assenting, and conveniently spending some time with his regiment and in Paris, but accepting without protest the eighteenth-century customs even when at home. In July of the year 1744 they were again at Cirey, Voltaire, as usual, in bed and hard at work. He was fifty and the Marquise was thirty-eight. They went up to Paris the latter part of August, preparing for the festival of rejoicing at Louis XV's recovery from an illness and his return from the wars. There was also Voltaire's *Princess* that was soon to be produced, and this demanded his presence at the capital—a production that brought him much acclaim.

A biographer who, being a woman, may be thought to have understood the Marquise, writes that "at his side was the woman who was the aptest pupil of Maupertuis and almost the only other person in France who understood Newtonianism save Maupertuis himself, Voltaire, and one Clairaut. The rest of the world was Cartesian." This Clairaut is the "M. Cléraut" of the letter, the Alexis-Claude Clairaut, then thirty-one years old, who had just published his *Théorie de la figure de la terre* (Paris, 1743). Five years later, while she was getting help from him in her work on Newton, he excited the jealous rage of Voltaire, who broke open a door at Cirey as a result, and then rushed down the stairs followed by the most notable woman mathematician of France and by one of the best-known men in the same field of work. Voltaire soon had occasion to repent his anger in sackcloth and ashes, for the Marquise died in 1749, at the age of forty-three. One of her Paris acquaintances, a grande dame of the salons, with the heartlessness of the age, remarked that "to die in childbed at her age is to wish to make oneself peculiar: it is to pretend to do nothing like other people." Frederick the Great, who would sacrifice all delicacy of feeling, if he ever had any, for the sake of a *bon mot*, suggested as her epitaph, "Here lies, one who lost her life in giving birth to an unfortunate infant and a treatise on philosophy." Thus came into light du Chastellet's work on Newton, together with a child of uncertain parentage, and thus passed away the most brilliant woman of both a country and a century of women of exceeding brilliancy,—intellectually, socially, and, probably in more cases than society history would lead us believe, morally as well.

13. Cassini Completes the Great Survey of France.

The first noteworthy attempt at measuring the earth, made in modern times was that of Jean Fernel, about 1528. He took the arc determined by Paris and Amiens, the two stations being located approximately on the same meridian. Considering the instruments available, his results were remarkable. He found that $1° = 57{,}099$ toises, while the mean obtained by Lacaille and Delambre in the latter part of the eighteenth century, at which time the instruments were very satisfactory, was 57,068 toises. In 1669 and 1670 Jean Picard carried on an elaborate system of triangulation and found that $1° = 57{,}060$ toises. This meridian was extended by Jean-Dominique Cassini (Cassini I) in 1701 and his work was continued by his son, Jacques, the results being published in Paris in 1720 and serving to set on foot the elaborate surveys which finally determined the spheroidal shape of the earth. César François Cassini de Thury, the son of Jacques, carried on the work of his father and grandfather, publishing his results in 1740, and in 1744 he announced his plan for extending the work to include the making of a map of France. He was followed by a representative of the fourth generation of the remarkable Cassini family, Jacques-Dominique, who continued the surveys for the map. In 1793 the work appeared in 180 sheets, but there still remained much to be done, and it was not until June 19, 1803, that the last of the four Cassinis was able to report to the authorities that the

great work, which may be said to have begun indirectly in 1528, and directly in 1744, was at last finished. The letter announcing this fact is in my collection and reads as follows:

To General Sanson, Director of the Depôt of War:

Count Cassini, Member of the Institute, of the General Council, and of the Electoral College of the Department of the Loire:

General, I venture to flatter myself that, under the direction of a brave and loyal soldier like yourself, I can see, at the General Depôt of War, the end of the interminable affair of the Carte Générale de la France.

You have before you or your representative the latest decree of the Council of State that governs the acts of those engaged upon the map of France. I have examined carefully the report which they have made, covering the two years under the Minister of War, and I am returning it with the recommendation that a general settlement be made in accordance therewith.

This report strengthens still more the agreements between the minister and those who have acted for the Compagnie de la Carte Générale de la France, concerning the distribution of the maps to the subscribers and to the associates.

The decree places the matter hereafter in your hands. It is for the Depôt of War to cancel the various obligations, and now it becomes the real owner of the material which has been prepared under the act relating thereto. . . .

I cannot retire from the work without taking the opportunity of assuming the honor of offering to you the testimony of the distinguished consideration which you have inspired within me and which I shall retain as long as I live.

General,

Your very humble
CASSINI
Member of the Institute.

1st Messidor, an XI (June 19, 1803).

The letter bears the usual memoranda of reference for report, and of filing. The omitted portion has no special interest, referring only to details of distribution.

Thus ended a great scientific work that had extended over so many years,—the greatest work of its kind that had ever been undertaken up to that time, now more than a century ago.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

DISCUSSIONS.

The first discussion was suggested by Problem 2863 [*1920*, 482];[1] it appears in this department because the method of attack and the incidental results give it an independent interest. Mr. T. L. Bennett discusses certain properties of roulette curves by the method of circular coördinates. He has presented the material in such form as to imply no previous acquaintance with this method; and indeed his discussion gives an adequate introduction to the study of the properties of circular coördinates.[2]

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[1] See the problem department of this issue of the MONTHLY.

[2] The idea of circular coördinates seems to have originated with Plücker. Some articles dealing with the subject are: "Ueber Kreiscoordinaten" by W. Stammer, *Journal für die reine und angewandte Mathematik*, vol. 44, 1852, pp. 295–316; "On some applications of circular coordinates" by F. Franklin, *American Journal of Mathematics*, vol. 12, 1890, pp. 161–190; "Sundry metric theorems concerning n lines in a plane" by F. H. Loud, *Transactions of the American Mathematical Society*, vol. 1, 1900, pp. 323–338; and F. Morley's article, pp. 97–115, referred to below. —EDITOR.

Mr. R. S. Underwood proposes and answers, in the second discussion, an interesting question concerning trigonometric functions: when will the sine or cosine of an angle rationally expressible in degrees be rational? It may be remarked that it is well known that $\sin\theta$ and θ, when θ is expressed in radians, can never simultaneously be even algebraic, much less rational numbers, for $\theta \neq 0$, and that a similar statement holds for each of the other elementary trigonometric functions. Proofs of these theorems are somewhat intricate and are closely related to the famous proofs of the transcendentality of e and π. With the same notation, Mr. Underwood's theorem may be said to deal with the simultaneous rationality of θ and $\sin\pi\theta$. This belongs to a simpler order of ideas and may be settled by an easy and direct method of attack. It has perhaps a more obvious interest for the field of collegiate mathematics than the more difficult question mentioned above. Mr. Underwood leaves open the nature of the result for the tangent and cotangent; without doubt the facts can be ascertaine sid bymilar methods in these cases.

I. A Theorem on Hypocycloids, by the Method of Circular Coördinates.

By T. L. Bennett, University of Illinois.

Let the plane be referred to rectangular coördinates (X, Y). The circular coördinates (x, y) of the point (X, Y) are defined to be $x = X + iY$, $y = X - iY$.

A complex number $a + ib$, for which $a^2 + b^2 = 1$, is called a *turn*. A turn may be written $e^{i\theta}$, where $\theta = \cos^{-1} a = \sin^{-1} b$. It is easily shown that the conjugate of a turn is its reciprocal, and that any product of turns is a turn. All points which have turn coördinates lie on a unit circle about the origin. A turn will be denoted by some form of the letter t.

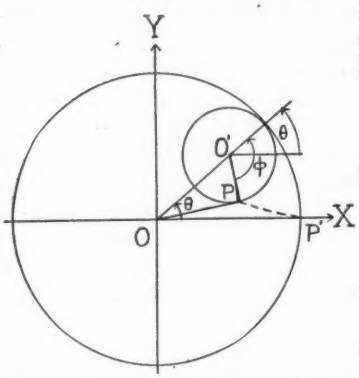
A curve may be represented by a map equation, $x = f(t)$. When t assumes turn values we get values of x, thus determining the points of the curve, because y is known as soon as x is known, being the conjugate of x. If $f(t)$ is a rational function, there is thus established a definite correspondence between the points of the unit circle and the points of the curve. If in the map equation t is replaced by any function of t which is a turn when t assumes turn values, the curve is not changed, as we have merely a different distribution of the parameter along the curve. With every map equation, $x = f(t)$, is associated another equation, namely, $\bar{x} = y = \overline{f(t)}$. If t be eliminated between these two equations, the result is the equation of the curve in circular coördinates.

The equation in circular coördinates of any straight line may be written in the form $t_1 x + y = rt_1$. The coefficient t_1 is called the *clinant* of the line, and determines the direction of the line thus: the angle which the line makes with the positive real axis is found to be

$$\phi = \frac{-1}{2i} \log(-t_1).$$

If a curve is defined by the map equation $x = f(t)$, it can be shown that the clinant of the tangent at any point is $-(dy/dt)/(dx/dt)$.

Let OX and OY be the real and imaginary axes, respectively, and let the circle O', of radius r_2, roll inside the circle O, of radius r_1, with $r_1 > r_2$, and r_1/r_2 rational. Any point on the circle O', as P, will describe a hypocycloid. Let the initial position of P be P', on OX. The map equation of the locus of P is obtained as follows:



$$\text{vector } OP = \text{vector } OO' + \text{vector } O'P,$$

$$x = (r_1 - r_2)e^{i\theta} + r_2 e^{i(\theta-\phi)}.$$

Since $r_1\theta = r_2\phi$, this equation becomes

$$x = (r_1 - r_2)t + r_2 \cdot \frac{1}{t^{\frac{r_1-r_2}{r_2}}},$$

where $t = e^{i\theta}$.

As a consequence of this equation it easily follows that if $r_1/r_2 = p/q$, the cycloid so generated will be identical with the cycloid for which $r_1/r_2 = p/(p-q)$. This means that if a circle of radius $r_1 - r_2$ be rolled inside the circle O, with the initial point of contact at P', the hypocycloid so generated will be identical with that generated by P.

If $r_1/r_2 = 2/1$, we have $x = r_2[t + (1/t)]$, which is the equation of the line segment of length $4r_2$ lying along the real axis, having its center at the origin. It will be found convenient to extend this equation as follows: if a and b are any complex numbers, then from the graphical addition and multiplication of complex numbers the following facts are obvious: $x = a + b[t + (1/t)]$ is the map equation of a line segment with center at $x = a$, of length $4|b|$, inclined to the positive real axis at the angle amp b, with extremities at the points for which t equals 1 and -1.

We shall now demonstrate the following theorem: *For any odd prime p, the $\frac{1}{2}(p-1)$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.*

Let $r_1/r_2 = p/q$. Since p is prime, and $q < p$, we may take $r_1 = p$, and $r_2 = q$. Now as p remains fixed, and q takes on all integral values from 1 to $p-1$ inclusive, there are generated $p-1$ cycloids. But from a preceding theorem it is seen that only half of them are distinct, and these may be generated by q taking the values from 1 to $\frac{1}{2}(p-1)$ inclusive. Hence we shall further restrict q so that $q < \frac{1}{2}p$. The cycloid for which $q = k$ will be called, for brevity, the curve $q = k$.

The general curve of the system is

$$x = (p-q)t + \frac{q}{t^{\frac{p-q}{q}}} = (p-q)T^q + \frac{q}{T^{p-q}},$$

where $t = T^q$. For any particular value of q, as k, this equation is

$$x = (p-k)T^k + \frac{k}{T^{p-k}}.$$

Consider the curve[1]

$$x = T_1^k(p - 2k) + kT_1^{k-1}T + \frac{k}{T_1^{p-k-1}T}.$$

It is clear that it cuts the curve $q = k$ at the point $T = T_1$. But by computing the values of $-(dy/dt)/(dx/dt)$ for this new curve and the curve $q = k$, it is seen that the curves are tangent at the point $T = T_1$, because at this point the clinants of the two curves are the same. This new equation may be written

$$x = T_1^k(p - 2k) + \frac{k}{T_1^{\frac{p}{2}-k}}\left[T_1^{\frac{p-2}{2}}T + \frac{1}{T_1^{\frac{p-2}{2}}T}\right].$$

Hence this is the equation of a line segment of length

$$4\left|\frac{k}{T_1^{\frac{p}{2}-k}}\right| = 4k,$$

with extremities given by $T_1^{(p-2)/2}T = \pm 1$, that is, at the points

$$x = T_1^k(p - 2k) \pm \frac{2k}{T_1^{\frac{p}{2}-k}}.$$

These two points lie on the hypocycloid

$$x = (p - 2k)T^{2k} + \frac{2k}{T^{p-2k}}, \quad [\text{for } T = \pm\sqrt{T_1}],$$

which is either the equation of the curve $q = 2k$, or the equation of the curve $q = p - 2k$, according as $2k$ is or is not less than $\frac{1}{2}p$.

Hence, if k be any positive integer less than $\frac{1}{2}p$, if a segment of length $4k$ be suitably taken on the tangents to the curve $q = k$, the extremities of these tangents lie either on the curve $q = 2k$ or on the curve $q = p - 2k$, as above indicated. It is then clear that the successive values of q for the sequence of curves of the theorem must be obtained by successive doubling,[2] by then reducing mod p to values between $-\frac{1}{2}p$ and $\frac{1}{2}p$, and then taking the absolute value of the result.

That the curves may be arranged in cycles follows from Fermat's Theorem: $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$; whence $k2^{(p-1)/2} \equiv \pm k \pmod{p}$. There will be one or more cycles according as $\frac{1}{2}(p - 1)$ is or is not the smallest integer which will satisfy the congruence $k2^x \equiv \pm k \pmod{p}$. For example, if $p = 13$, the values of q form the single cycle 4 5 3 6 1 2, while for $p = 17$ we get the two cycles 3 6 5 7 and 8 1 2 4.

The proof for $p = 5$ is included in the foregoing as a special case.

[1] Professor Morley has called such curves "penosculants" of the original curve. See his paper "On the metric geometry of the plane n-line," *Trans. of the Amer. Math. Soc.*, vol. 1, p. 102.

[2] In the general equation of the hypocycloid it was assumed that $r_1 > r_2$. It may be readily shown that if $r_1 < r_2$, this equation represents an epicycloid, which is identical with the epicycloid generated by rolling a circle of radius $r_2 - r_1$ on the outside of the circle O, with the initial point of tangency at P'. Therefore the above analysis shows that if we start with any positive integer for q, (prime to p), then by successive doubling of q, without reducing mod p, we obtain an infinite sequence of epicycloids having the property mentioned in the theorem.

II. On the Irrationality of Certain Trigonometric Functions.

By R. S. Underwood, Purdue University.

Theorem: *If an angle, expressed in degrees, is rational and not a multiple of* 30°, *its sine, cosine, secant, and cosecant are irrational.*

By De Moivre's Theorem,

$$\cos n\theta + i \sin n\theta = (\cos \theta + i \sin \theta)^n. \tag{1}$$

Equating $\cos n\theta$ to the real part of the binomial expansion, we get

$$\cos n\theta = \cos^n \theta - \frac{n(n-1)}{2!} \cos^{n-2} \theta \sin^2 \theta + \frac{n(n-1)(n-2)(n-3)}{4!} \cos^{n-4} \theta \sin^4 \theta - \cdots. \tag{2}$$

(The letters m and n stand for integers throughout this discussion. For convenience we shall call an angle integral, rational, or irrational in accordance with the character of its value expressed in degrees.)

From (2) it appears that $\cos n\theta$ can always be expressed in terms of integral powers of $\cos \theta$, and that $\cos \theta$ is irrational if $\cos n\theta$ is irrational. Since $\cos (45° \pm m180°)$ is irrational, $\cos \dfrac{m180° \pm 45°}{45}$, or $\cos (4m \pm 1)°$ is irrational. Every odd number can be expressed in the form $(4m \pm 1)$, and therefore the cosine of every odd integral angle is irrational.

Testing the equation

$$\sin 30° = \sin 10°(3 - 4 \sin^2 10°) = 1/2$$

for rational roots, we find that $\sin 10°$ is irrational. Since $\cos 30°$ and hence $\cos 10°$ is irrational, it follows that the cosines of $(100° \pm m180°)$, $(80° \pm m180°)$, and $(40° \pm m180°)$, and hence of $(9m \pm 5)°$, $(9m \pm 4)°$, and $(9m \pm 2)°$, are irrational. The last angle includes the form $(9(2s) \pm 2)°$; therefore, $\cos (9m \pm 1)°$ is irrational. The only even numbers which cannot be expressed in these forms are multiples of 6. By a geometric method we can show that $\sin 18° = (\sqrt{5} - 1)/4$ and $\cos 18° = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}$ are irrational, and hence this is true also of the cosines of 108°, 72°, 54°, 36°, 24°, 12° and 6°. Then, since $\cos 12°$ is irrational, $\sin 6°$ (or $\sqrt{[1 - \cos 12°]/2}$) is irrational; and we can add the cosines of 96°, 84°, 48°, 42°, $66° = (180° - 48°)/2$, and $78° = (180° - 24°)/2$ to the list. Thus, in general, when n is not a multiple of 30°, $\cos n°$, and obviously $\sin n°$ as well, is irrational.

Furthermore, $\cos (m°/n)$ when m/n is a rational non-integral fraction reduced to its lowest terms and m is not a multiple of 30°, is evidently irrational since $\cos m°$ is irrational. For every such angle $m°/n$ there is a complementary angle $h°/n$ $(h \neq k30)$ whose sine is irrational, and vice versa; hence both the sine and the cosine of the general rational angle $m°/n$ $(m \neq k30)$ are irrational.

To investigate an angle of the form m/n when m is a multiple of 30 and prime

to n (which is therefore odd), we may use the following expansion of $\sin n\theta$, valid when n is odd:[1]

$$\sin n\theta = n \sin\theta - \frac{n(n^2-1^2)}{3!}\sin^3\theta + \frac{n(n^2-1^2)(n^2-3^2)}{5!}\sin^5\theta - \cdots (-1)^{(n-1)/2}2^{n-1}\sin^n\theta. \tag{3}$$

The coefficients in this expansion are integers, as may be seen by equating the imaginary parts in the expansion of (1).

The coefficient of the $(r+1)$th term in (3) contains the factor 2^{2r}. For, upon writing this coefficient,

$$\frac{(n-2r+1)\cdots(n-3)(n-1)n(n+1)(n+3)\cdots(n+2r-1)}{(2r+1)!},$$

we see that in the $2r$ successive even numbers contained in the numerator, there will be

$2r$ numbers containing the factor 2,
r numbers containing the factor 2^2,
at least $[r/2]$ numbers containing the factor 2^3,
at least $[r/2^2]$ numbers containing the factor 2^4,
. . . . . .
at least $[r/2^\mu]$ numbers containing the factor $2^{\mu+2}$.

By $[r/2]$ is meant the largest integer in $r/2$. Choose μ so that $[r/2^\mu] = 1$. The sum of the exponents of 2 in the numerator will be *at least*

$$2r + r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

In $(2r+1)!$ the sum of the exponents of 2 is *exactly*

$$r + \left[\frac{r}{2}\right] + \left[\frac{r}{2^2}\right] + \cdots + \left[\frac{r}{2^\mu}\right].$$

Hence the excess of exponents of 2 in the numerator will be at least $2r$, and the coefficient will contain the factor 2^{2r}.

Letting $2r+1 = n$, we see that the exponent of 2 in the coefficient of $\sin^n\theta$ is no greater than the required $2r$. Then we can write

$$2^{n-1}\sin^n\theta - 2^{n-3}k_1\sin^{n-2}\theta + 2^{n-5}k_2\sin^{n-4}\theta - \cdots \pm n\sin\theta = \pm\sin n\theta.$$

If we put

$$x = 2\sin\theta,$$

then

$$x^n - k_1x^{n-2} + k_2x^{n-4} - \cdots \pm nx = \pm 2\sin n\theta.$$

All the coefficients in this equation except the absolute term on the right are integers. Let $\theta = m^\circ/n$, where m is an odd multiple of 30 or a multiple of 180, and prime to n. Then $2\sin n\theta = 2\sin m^\circ = 0, \pm 1, \pm 2$, and the absolute term is also an integer. Therefore the rational roots are integers, and $\sin\theta$ is half of an integer. Then $\sin\theta = 0, \pm\frac{1}{2}$, or ± 1, which is impossible.

[1] Loney: *Analytical Trigonometry*, p. 69.

This disposes of every case except when m is a multiple of 60 but not of 180. In this case $\sin n\theta = \pm \frac{1}{2}\sqrt{3}$, and by (3), $\sin\theta$ is irrational.

It has now been shown that the sine of every rational angle not a multiple of 30° is irrational. Obviously this establishes the theorem for the cosine, secant, and cosecant functions as well.

An analogous theorem, differing only in the substitution of 45° for 30°, may probably be shown to hold true for the tangent and cotangent.

RECENT PUBLICATIONS

REVIEWS

AMERICAN MEN OF SCIENCE.

American Men of Science. A Biographical Directory. Edited by J. McK. CATTELL and D. R. BRIMHALL. Third edition. Garrison, N. Y., The Science Press, 1921. 4to. 8 + 808 pp. Price $10.00.

The first edition of this work, published early in 1906, contained brief sketches of about 4,000 living American[1] men and women of science, the second edition, published late in 1910, about 5,500; the present edition contains some 9,500 sketches. The increase in the number of sketches measures roughly the increase of scientific workers.

In the first edition a star was prefixed to the subject of research in the case of one thousand of the sketches of students of the natural and exact sciences in the United States. In each of the twelve principal sciences[2] the names were arranged in the order of merit by ten leading students of the science. In this way the subjects of research of 80 mathematicians and 50 astronomers were starred.

In the second edition, the thousand leading men of science were determined in the same manner as in the first edition, stars being added to the subjects of research in the case of 269 new men; of these 20 were mathematicians. Their names, the names of the original 80 mathematicians, as well as of the 29 new mathematicians whose subjects of research are starred in the present edition, are given below.

It will be observed that in all editions the total number of sketches, in connection with which mathematics is starred, is 129. Of these, 11 refer to those who have died. There remain 118 living mathematicians with a star in the present edition, since a star once given in a sketch is not removed in subsequent editions. Hence 38 of these mathematicians are now no longer regarded as among the 80

[1] This term is interpreted as applying not only to natives of the United States and of the Dominion of Canada, but also to foreigners temporarily resident in these countries; for example: O. Bolza and P. Boutroux.

[2] Mathematics, physics, chemistry, astronomy, geology, botany, zoölogy, physiology, anatomy, pathology, anthropology, and psychology.

to be counted in the thousand leading men of science. The names of 29 of these 80 are, as already indicated, given below; but the membership of the remaining particular group of 51 is, except to the editors, unknown.

The names of those for whom mathematics is starred are as follows: R. E. Allardice (2, 3), R. C. Archibald (3), H. Bateman (3), W. W. Beman (1, 2, 3), A. A. Bennett (3), G. D. Birkhoff (2, 3), E. M. Blake (1, 2, 3), H. F. Blichfeldt (1, 2, 3), G. A. Bliss (2, 3), H. Blumberg (3), M. Bôcher (1, 2; d. 1918), O. Bolza (1, 2, 3), C. L. Bouton (1, 2, 3), E. W. Brown (1, 2, 3), W. E. Byerly (1, 2, 3), F. Cajori (1, 2, 3), R. D. Carmichael (3), A. S. Chessin (1, 2, 3), E. W. Chittenden (3), A. B. Coble (2, 3), F. N. Cole (1, 2, 3), L. L. Conant (1, 2; d. 1916), J. L. Coolidge (1, 2, 3), D. R. Curtiss (3), E. W. Davis (1, 2; d. 1918), L. E. Dickson (1, 2, 3), L. W. Dowling (2, 3), A. Dresden (3), W. H. Echols (1, 2, 3), H. T. Eddy (1, 2, 3) [Math., Mech., Eng.], J. A. Eiesland (2, 3), L. P. Eisenhart (2, 3), A. Emch (1, 2, 3), G. C. Evans (3), H. B. Fine (1, 2, 3), T. S. Fiske (1, 2, 3), W. B. Fite (1, 2, 3), W. B. Ford (2, 3), F. Franklin (1, 2, 3), E. Frisby (1, 2, 3) [Math., Ast.], O. E. Glenn (3), W. C. Graustein (3), T. H. Gronwall (3), J. G. Hagen (1, 2, 3) [Math., Ast.], G. B. Halsted (1, 2, 3) [Math., Logic], H. Hancock (1, 2, 3), M. W. Haskell (1, 2, 3), C. N. Haskins (2, 3), A. S. Hathaway (1, 2, 3) [Math., Physics], H. E. Hawkes (1, 2, 3), E. R. Hedrick (1, 2, 3), T. H. Hildebrandt (3), T. F. Holgate (1, 2, 3), L. S. Hulburt (1, 2, 3), E. V. Huntington (1, 2, 3), W. A. Hurwitz (3), J. I. Hutchinson (1, 2, 3), E. W. Hyde (1, 2, 3), D. Jackson (3), W. W. Johnson (1, 2, 3), L. C. Karpinski (3), E. Kasner (1, 2, 3), O. D. Kellogg (2, 3), C. J. Keyser (1, 2, 3), S. Lefschetz (3), D. N. Lehmer (2, 3), G. H. Ling (1, 2, 3), E. O. Lovett (1, 2, 3) [Math., Celestial mech.], P. Lowell (1, 2; d. 1916) [Math., Orientalism, Ast.], A. C. Lunn[1] (2, 3), E. McClintock (1, 2; d. 1916) [Math., Actuarial science], J. Maclay (1, 2; d. 1919), J. McMahon (1, 2, 3) [Math., Mathematical physics], W. D. MacMillan (3) [Math., Ast.], H. P. Manning (1, 2, 3), W. A. Manning (2, 3), H. Maschke (1; d. 1908), M. Mason (2, 3), W. H. Metzler (1, 2, 3), G. A. Miller (1, 2, 3), H. H. Mitchell (3), C. L. E. Moore (3), C. N. Moore (3), E. H. Moore (1, 2, 3), R. L. Moore (3), F. Morley (1, 2, 3), F. R. Moulton (1, 2, 3) [Math., Celestial mech.], H. B. Newson (1; d. 1910), W. F. Osgood (1, 2, 3), J. M. Page (1, 2, 3), J. M. Peirce (1; d. 1906), A. Pell (1, 2, 3; d. 1921), Mrs. A. Pell (3), J. Pierpont (1, 2, 3), M. B. Porter (1, 2, 3), A. Ranum (2, 3), R. G. D. Richardson (3), H. L. Rietz (2, 3), E. D. Roe (1, 2, 3) [Math., Ast.], W. H. Roever (3), P. L. Saurel (1, 2, 3) [Math., Physics[2]], Charlotte A. Scott (1, 2, 3), F. R. Sharpe (3), J. B. Shaw (1, 2, 3), C. H. Sisam (3), C. S. Slichter (2, 3), D. E. Smith (1, 2, 3), P. F. Smith (1, 2, 3), W. B. Smith (1, 2, 3) [Math., New testament criticism[2]], V. Snyder (1, 2, 3), H. F. Stecker (1, 2, 3), W. E. Story (1, 2, 3), W. I. Stringham (1; d. 1909), H. Taber (1, 2, 3), J. H. Tanner (1, 2, 3), H. D. Thompson (1, 2, 3), E. J. Townsend (3), H. W. Tyler (1, 2, 3), E. B. Van Vleck (1, 2, 3), O. Veblen (2, 3), J. H. Wedderburn (3), H. S. White (1, 2, 3), J. K. Whittemore (2, 3), E. J. Wilczynski (1, 2, 3), E. B. Wilson

[1] In the third edition the subject for A. C. Lunn is given as mathematical physics.
[2] This subject does not occur in the third edition.

(1, 2, 3), F. S. Woods (1, 2, 3), J. W. A. Young (1, 2, 3) [Math., Pedagogy of math.], J. W. Young (2, 3), A. Ziwet (1, 2, 3) [Math., Mech.].

Fourteen names in this list are followed by brackets [ ] enclosing the names of different fields of work, but mathematics coming first. The following 10 names are of others in connection with whom mathematics does not come first or else the subject is mathematical astronomy or mathematical physics:

J. G. Coffin (2, 3) [Mathematical physics], C. L. Doolittle (1, 2; d. 1919) [Ast., Math.], J. R. Eastman (1, 2; d. 1913) [Ast., Math.], G. W. Hill (1, 2; d. 1914) [Mathematical astronomy], F. H. Loud (1, 2, 3) [Ast., Math.], A. Macfarlane (1, 2; d. 1913) [Mathematical physics], R. C. Maclaurin (2; d. 1920) [Mathematical physics], B. O. Peirce (1, 2; d. 1914) [Mathematical physics], F. Slate (1, 2, 3) [Mathematical physics], J. B. Webb (1, 2; d. 19 ?) [Physics, Math.].

On pages 771–780 is given a very useful list of 1059 American men of science who died between January 1, 1903, and December 31, 1920. The years of birth and death are appended in each case where it was possible to determine such dates. The list includes not only the names of those whose names appeared in earlier editions, but also of others such as of G. M. Green. Certain events of very recent occurrence are recorded in the volume; for example, the death of A. Pell, January 26, 1921, and the appointment of J. R. Angell as president of Yale. It is not clear to the reviewer why a sketch of W. G. Everett, whose subject is ethics, should appear in the third edition.

Everyone interested in American science will wish to have this new *Biographical Directory* constantly at hand.

R. C. ARCHIBALD.

NOTES.

School Arithmetics is the title of a work, for grades I–VIII, by G. WENTWORTH and D. E. SMITH. (3 books, Boston, Ginn, 1920. 12mo. $6 + 282 + 16$; $6 + 298 + 19$; $6 + 346 + 19$ pp. Price $72 + 76 + 92$ cents.)

Professor J. H. M. WEDDERBURN's article "On equations of motion of a single particle," published separately January 24, 1921, appeared in part 1 (May), volume 41 (pp. 26–33), of *Proceedings of the Royal Society of Edinburgh.*

In *Popular Astronomy*, June–July, 1921, there is a reproduction of a photograph, taken on May 6, 1921, of a group with Professor Einstein at the Yerkes Observatory. Professor A. C. LUNN, of the University of Chicago, is a member of the group.

Comptes Rendus du Congrès International des Mathématiciens à Strasbourg, was published November 1, 1921, by Imprimerie Edouard Privat, Toulouse (paper, 100 francs; cloth, 125 francs).

In *Proceedings of the London Mathematical Society*, second series, volume 20, August 20, 1921, there is an article "Arithmetic of quaternions" by L. E. DICKSON. Compare *1921*, 289.

The last number of *Mathematische Annalen*, volume 80, has been published (see *1921*, 219). It is a General register to volumes 51–80 and has a preface by H. VERMEIL. There is a portrait frontispiece of C. Neumann.

The third edition of W. F. Osgood's *Lehrbuch der Funktionentheorie*, volume 1, published by Teubner in 1920, is an anastatic reprint, the only textual changes of the second edition being in connection with the correction of a few misprints. The paper and general appearance are inferior and the volume is only about half as thick.

In the recently published *Washington University Studies*, volume 8, no. 2, the following papers are included: "The curve which with its caustic encloses the minimum area" (cf. this MONTHLY, *1920*, 225) by OTTO DUNKEL, 183–194; and "Brilliant point phenomena" (cf. this MONTHLY, *1913*, 299; *1919*, 111) by W. H. ROEVER, 131–159 + 4 plates.

Abhandlungen aus dem mathematischen Seminar der Hamburger Universität is the title of one of the nine new mathematical periodicals founded in 1921. The editors are W. BLASCHKE, E. HECKE and J. RADON. The first number (98 pages, price 25 marks) was published in September (Hamburg, Otto Meissner). It contains six memoirs by the following authors: W. Blaschke, E. Hecke, A. Ostrowski, J. Radon, K. Reidemeister, and P. Steinhagen.

We have previously noticed the *Tables of the Digamma and Trigamma Functions* by Miss ELEANOR PAIRMAN (*1921*, 265–266). During the summer of 1921 she completed the work for a doctorate at Harvard University, her thesis being entitled: "Expansion theorems for solutions of a Fredholm linear homogeneous integral equation of the second kind with kernel of special non-symmetric type."

A volume supplementary to *Œuvres de Fermat*, tomes I–IV, is being prepared —Heath, of New York, announces as in the press *Projective Geometry* by R. M. WINGER, of the University of Washington, and *Trigonometry, Plane and Spherical* by J. A. BULLARD and ARTHUR KIERNAN, of the United States Naval Academy— Wiley, of New York, published, in December, 1921, *First Course in the Theory of Equations* by L. E. DICKSON (171 pages; $1.75)—Macmillan of New York has published *Mathematics for Students of Agriculture* by S. E. RASOR.

The number of *L'Enseignement Mathématique*, published in July, 1921, contains a complete list of the publications of the International Commission on the Teaching of Mathematics. This list gives references to 310 reports published in 187 fascicules or volumes, and totalling 13,565 pages. To this total Germany contributed 5,571 pages, the United States coming next with 1,499 pages, then Great Britain with 921, Japan with 788, Switzerland with 781, Austria with 776, and similarly for twelve other countries down to Argentina with 24 pages and Roumania with 16. Among other indexes, an alphabetical list of authors is appended to the list.

In notes on perfect numbers, we have indicated (*1921*, 140–141) that the Mersenne numbers, $p = 2^n - 1$, are prime for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107$, and 127; that it was not known whether p was prime or composite for $n = 137, 139, 149, 157, 167, 193, 199, 227, 229, 241$, and 257; and that for all other values of $n < 257$, p was composite. A. Gérardin has indicated a method (*Comptes rendus du Congrès des Sociétés Savantes en 1920*, Sciences, pp. 53–55) showing that p is composite for $n = 137$. Hence there are now only 10 doubtful cases, $n < 257$, in connection with the Mersenne numbers.

Among parts of *Encyklopädie der mathematischen Wissenschaften* published in 1921 are the following: (*a*) II, 3, 4—"Neuere Untersuchungen über Funktionen von komplexen Variabeln" by L. Bieberbach, 379–532 (22 marks); (*b*) III, 1, 6—"Elementargeometrie und elementare nichteuklidische Geometrie in synthetischer Behandlung" (Teil 2, Schluss) by M. Zacharias, 963–1172 (23.75 marks); (*c*) III, 1, 7—"Neuere Dreiecksgeometrie" by G. Berkhan and F. Meyer, and "Systeme geometrischer Analyse" by H. Rothe, 1173–1423 (30.80 marks); (*d*) III, 2, 7—"Mehrdimensionale Räume" by G. Segre, 769–972 (22 marks); (*e*) III, 3, 5—"Dreifach orthogonale Flächensysteme" by E. Salkowski, 541–606 (9.70 marks); (*f*) II, 2, 5—"Nichtlineare Differentialgleichungen" by E. Hilb, and "Abelsche Funktionen und allgemeine Thetafunktionen" by W. Wirtinger, and Titel, Inhaltsverzeichnis und Register zu Band II, 2, 563–897 (49.30 marks); (*g*) II, 3, 5—"Arithmetische Theorie der algebraischen Funktionen" by K. Hensel and "Arithmetische Theorie der algebraischen Funktionen zweier unabhängigen Veränderlichen" by H. W. E. Jung, 533–674; (*h*) V, 1, 6—"Physikalische- und Elektrochemie" by K. F. Herzfeld and Titel und Inhaltsverzeichnis zu Band V, 1, 947–1112 + 20 (30 marks); (*i*) V, 2, 4—"Relativitätstheorie" by W. Pauli, Jr., 539–775 (36 marks).

The first number (80 pages, sm. 4to) of a new periodical, *Zeitschrift für angewandte Mathematik und Mechanik*, was published in February, 1921, by the Verlag des Vereines Deutscher Ingenieure, Berlin (Price, in Germany, 50 marks; to members of the Deutsche Mathematiker-Vereinigung in America, 72 marks a year for the six numbers). The periodical is edited by R. von Mises with the assistance of A. Föppl, G. Hamel, R. Mollier, H. Müller, L. Prandtl, and R. Rüdenberg. The first number opens with an article by the editor: "Zur Einführung, Ueber die Aufgaben und Ziele der angewandten Mathematik" (pp. 1–15). This is followed by a variety of material under the headings: Principal papers, Comprehensive reports, Short abstracts, Book-reviews, Short notices, and News.

Journal of Mathematics and Physics of the Massachusetts Institute of Technology is the title of a new periodical, the first number of which was published in November, 1921 (The Murray Printing Co., Cambridge, Mass.). The editors are F. S. Woods (professor of mathematics), H. M. Goodwin (professor of physics and electro-chemistry), and F. G. Keyes (professor of physico-chemical research); the managing editor is C. L. E. Moore (professor of mathematics). A volume

of 200–250 pages (price $3.00) consisting of three or four numbers will probably be published each year. The contents of the current number are: "A new vector method in integral equations" by F. L. Hitchcock and Norbert Wiener, 1–20; "On the geometry of motion in curved n-space" by Joseph Lipka, 21–41; "The equation of state, with applications to viscosity" by H. B. Phillips, 42–53; "Some hydrodynamic aspects of group theory" by S. D. Zeldin, 54–62.

Among the articles in the concluding number of *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 1920, are the following: "Zur Biographie Johann Bolyais" by S. Wiener, 130–135 [new material concerning one of the founders of non-euclidean geometry while he was at the academy for engineers, and in military service]; "Die relative und absolute Bewegung bei Huygens" by J. A. Schouten, 136–144 [interesting for students of the history of relativity]; "Jan Versluys" by L. Schlesinger, 236–237 [Netherland mathematician, born 1845, died January 15, 1920. Among his publications listed by Schlesinger are five prize memoirs, crowned by the Mathematical Society of Amsterdam, and numerous books, some of which were widely used in schools of the Netherlands. Among his books are: *Leerboek der Stereometrie, Handboek der Meetkunde, Methoden bij het oplossen van Meetkundige Vraagstukken, Isometrie en Axonometrie, Inleiding tot de nieuwere Meetkunde van den driehoek*, and *Zes en negentij bewijzen voor het theorema von Pythagoras*. There are many books by Versluys not listed by Schlesinger].

In the new volume of the great *Encyclopædia of Religion and Ethics*, edited by James Hastings (volume XI, Sacrifice–Sudra, Edinburgh, T. & T. Black, 1921), there are articles (1) on "Science" by J. A. Thomson, regius professor of natural history at the University of Aberdeen (pages 252–261); and (2) on "Space" by G. J. Stokes, professor of philosophy and jurisprudence in University College, Cork, Ireland (pages 759–764). The sub-headings for (1) are: Definition and characteristics, aim of science, the scientific mood, the methods of science, scope of science, classification of sciences, correlation of sciences, limitations of science, science and feeling, science and philosophy, and science and religion. For (2) the sub-headings are: History, (*a*) Greek philosophy, (*b*) middle ages, (*c*) Cartesian and English philosophy, (*d*) Kant; Laws of thought and the concept of space. The article concludes with an acknowledged extract of three paragraphs from Professor Stokes's article, "The theory of mathematical inference," contributed to this MONTHLY, *1900*, 1–8.

While the last part of *Acta Mathematica*, volume 42, appeared before the close of 1920, no part of volumes 38 and 39 had then been distributed. The first of these volumes, in memory of Henri Poincaré, has now been published. It is a volume of 402 pages (price 50 Swedish kronor; on extra fine, thick, strong paper, 75 kronor). The contents are as follows: "Au lecteur," pages 1–2; "Henri Poincaré, Analyse de ses travaux scientifiques," 3–135 [revised by the author];

"Rapport sur les travaux de M. Cartan" by H. Poincaré, 137–145; "Lettres à M. Mittag-Leffler" by H. Poincaré, 147–160; "Lettres à M. Mittag-Leffler concernant le mémoire couronné du prix de S. M. le roi Oscar II" by H. Poincaré, 161–173; "Lettres à L. Fuchs" by H. Poincaré, 175–184; "Briefe an H. Poincaré" by L. Fuchs, 185–187; "Henri Poincaré, en mathématiques spéciales à Nancy" by P. Appell, 189–195; "Lettres à M. Mittag-Leffler" by P. Boutroux, 197–201; "L'œuvre mathématique de Poincaré" by J. Hadamard, 203–287; "Die Bedeutung Henri Poincaré's für die Physik" by W. Wien, 289–291; "Deux mémoires de Henri Poincaré sur la physique mathématique" by H. A. Lorenz, 293–308; "L'œuvre astronomique d'Henri Poincaré" by H. v. Zeipel, 309–385; "Henri Poincaré und die Quanten-theorie," 387–397; "Henri Poincaré" by P. Painlevé, 399–402.

"Anaximander's Book, the Earliest Known Geographical Treatise" is the title of an interesting fifty-page monograph, by W. A. HEIDEL, professor of Greek in Wesleyan University, published in *Proceedings of the American Academy of Arts and Sciences*, April, 1921. The mathematician will naturally wish to read collaterally what Sir Thomas Heath records concerning Anaximander in numerous passages of his *Aristarchus of Samos* (Oxford, 1913). From this latter source the following paragraphs, stripped of foot-note references, are extracted (pages 38–39):

"The story that Anaximander was the first to discover the *gnomon* (or sun-dial with a vertical needle) is incorrect, for Herodotos says that the Greeks learnt the use of the *gnomon* and the *polos* from the Babylonians. Anaximander may, however, have been the first to 'introduce' or make known the *gnomon* in Greece, and to show on it 'the solstices, the times, the seasons, and the equinox.' He is said to have set it up in Sparta. He is also credited with constructing a sphere to represent the heavens, as was Thales before him.

"But Anaximander has yet another claim to undying fame. He was the first who ventured to draw a map of the inhabited earth. The Egyptians had drawn maps before, but only of particular districts; Anaximander boldly planned out the whole world with 'the circumference of the earth and of the sea.' Hecataeus, a much-travelled man, is said to have corrected Anaximander's map, so that it became the object of general admiration. According to another account, Hecataeus left a written description of the world based on the map. In the preparation of the map Anaximander would of course take account of all the information which reached his Ionian home as the result of the many journeys by land and sea undertaken from that starting-point, journeys which extended to the limits of the then-known world; the work involved of course an attempt to estimate the dimensions of the earth. We have, however, no information as to his results."

Periodical Bibliographies and Abstracts for the Scientific and Technological Journals of the World is the title of a *Bulletin* (pages 131–154) published by the National Research Council in June, 1920. If ignorance of the subject in question is as glaringly in evidence in other topics as it is in connection with mathematics (page 145), the Council had been better advised to have withheld the manuscript of its *Bulletin* until it had been properly prepared—*Funds Available in 1920 in the United States for the Encouragement of Scientific Research* is the subject of an interesting 81-page *Bulletin* of the Research Council published in March, 1921. The grant of $25,000 from the General Education Board to the Mathematical Association of America is noted. Under the heading of "Mathematics" reference

is made also to the Sylvester prize of Johns Hopkins University, to the J. S. K. mathematical fellowship of Princeton University, and to the resident research fellowship in mathematics of Bryn Mawr College. Under the heading of "Science" there is a long list of medals and prizes, grants, institutional funds, and fellowships and scholarships, which are unrestricted in award. We have already referred to the Heckscher Foundation here listed, and to awards made from it for research in mathematics (*1921*, 287–288).

Among books published in 1921 are the following: *Theoretical Mechanics: An Introductory Treatise on the Principles of Dynamics* by A. E. H. LOVE (Third edition, Cambridge University Press; 15 + 310 pages; price 30 shillings)—*Elements of the Mathematical Theory of Electricity and Magnetism* by J. J. THOMSON (Fifth edition, Cambridge University Press; 410 pages; price 25 shillings)—*A Treatise on the Integral Calculus, with Applications, Examples, and Problems* by J. EDWARDS (London, Macmillan; volume 1, 907 pages; price 50 shillings)—*The Reign of Relativity* by Viscount HALDANE (London, Murray; 23 + 430 pages; price 21 shillings)—*A First Course in Statistics* by D. C. JONES (London, Bell; 9 + 286 pages; price 15 shillings)—*Des fondements de la géométrie* by H. POINCARÉ (Bibliothèque de synthèse scientifique. Paris, E. Chiron; 65 pages; price 3 francs)—*Theorie der reellen Funktionen* by H. HAHN (Berlin, Springer; volume 1, 4 + 600 pages; price 136 marks to Germans)—*Vorlesungen über Zahlen- und Funktionenlehre* by A. PRINGSHEIM. Band I: *Zahlenlehre* (Leipzig, Teubner, 3. [Schluss-] Abteilung[1]—*Komplexe Zahlen, Rechnen mit komplexen Gliedern;* price 175.00 marks to Germans)—*Vorlesungen über algebraische Geometrie* by F. SEVERI. Deutsch von E. Löffler (Leipzig, Teubner; 408 pages; price, unbound, 92.40 marks). Mathematisch-physikalische Bibliothek (Leipzig, Teubner, price of each, boards, 5.30 marks): no. 40, P. KIRSCHBERGER, *Mathematische Streifzüge durch die Geschichte der Astronomie* (54 pages); no. 42, A. WITTING, *Einführung in die Infinitesimalrechnung* 2. Auflage, II: *Die Integralrechnung* (50 pages)—*Gruppentheorie* by L. BAUMGARTNER (Sammlung Göschen, Berlin, Vereinigung wissenschaftlicher Verleger; 120 pages; price 5.00 marks to Germans). *Geschichte der Elementar-Mathematik* by J. TROPFKE, 2. vermehrte Auflage, Band 2: Die allgemeine Arithmetik (Berlin, V.W.V., 1921; 4 + 221 pages; price 50 marks to Germans).

The Institute of International Education was established in February, 1919, by the Carnegie Endowment for International Peace "to develop international good will by means of educational agencies, and for its specific purpose to act as a clearing house of information and advice for Americans concerning things educational in foreign countries and for foreigners concerning things educational in the United States." The administrative board selected to determine and guide the policy of the Institute consisted of "representatives of the endowed and the state universities, of the men's and women's colleges, and of international scholarship, law, finance, commerce, medicine, and journalism."

[1] Abteilung 1—*Reelle Zahlen und Zahlenfolgen* (1916, 12 + 292 pages; price 33.50 marks), Abteilung 2—*Unendliche Reihen mit reellen Gliedern* (1916, 8 + 22 pages; price 31.00 marks).

One of the results of the Institute's organization has been the arrangement for visits to this country of many foreign scholars and professors, and for our professors to lecture in foreign universities. Since the Institute pays travelling expenses both ways the arrangement may obviously be a happy one for those on sabbatic leave. In 1920–21 grants were made to 17 American professors to lecture in 9 countries: Argentina, Bohemia, China, England, France, Greece, Italy, Spain, and Turkey. The only mathematician was Professor SOLOMON LEFSCHETZ, of the University of Kansas, who lectured at the University of Rome. The Institute's *Annual Report*, of February 15, 1921, states that Doctor E. A. HORNE, professor of mathematics at the University of Patna, Patna, India, has been invited to this country by Harvard University.

Some publications of the Institute will be found invaluable for American students in certain foreign countries. We have already noticed the notable monograph of Kenneth McKenzie on *Opportunities for Higher Education in Italy* (61 pages; see *1919*, 300–301). There are also: *Opportunities for Higher Education in France* (148 pages), and G. E. MacLean's *Opportunities for Graduate Study in the British Isles* (40 pages). The *Guide Book for Foreign Students in the United States* (published July 1, 1921; 100 pages) will also be of interest to college teachers. Another similar bulletin, just published by the Bureau of Education, Washington (Bulletin, 1921, no. 6), is entitled: *Opportunities for Study at American Graduate Schools* (59 pages).

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 43, no. 2, April, 1921: "Boundary value and expansion problems: algebraic basis of the theory" by R. D. Carmichael, 69–101; "Algebraic theory of the expressibility of cubic forms as determinants, with application to Diophantine analysis" by L. E. Dickson, 102–125; "The impossibility of Einstein fields immersed in flat space of five dimensions" by E. Kasner, 126–129; "Finite representation of the solar gravitational field in flat space of six dimensions" by E. Kasner, 130–133; "On the motion of two spheroids in an infinite liquid along their common axis of revolution" by B. Datta, 134–142.

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, May, 1921: "The February meeting of the American Mathematical Society" by R. G. D. Richardson, 341–351; "Note on equal continuity" by J. F. Ritt, 351–353; "A new method in Diophantine analysis" by L. E. Dickson, 353–365; "The transformation of elliptic integrals" by J. H. McDonald, 366–373; "Bachmann on Fermat's last theorem" by H. S. Vandiver, 373–376 [Review of *Das Fermatproblem in seiner bisherigen Entwickelung* (Berlin and Leipzig, 1919)]; "Two books about airplanes" by E. W. Brown, 377–381 [Review of E. B. Wilson's *Aeronautics: A class text* (New York) and of H. G. Bader's *Grundlagen der Flugtechnik: Entwerfen und Berechnung von Flugzeugen* (Berlin)]; Review by F. M. Morgan of E. Beutel's *Die Quadratur des Kreises* (Leipzig and Berlin, 1920), 382; Notes, 383–386; New Publications, 386–388.

MATHEMATICAL GAZETTE, volume 10, March, 1921: "Address on relativity" by A. S. Eddington, 228–233; "The teaching of mathematics to boys whose chief interests are non-mathematical" by S. H. Clarke, 234–238; "The early history of the Association, or the passing of Euclid from our schools and universities and how it came about" by J. M. Wilson, 239–244; "Further reminiscences" by A. A. Bourne and F. S. Marshall, 244–247; "Mathematics in the lycées" by E. M. Read, 248–254—May, 1921: "Aeroplane mathematics" by S. Brodetsky, 257–281; "Gleanings far and near," 281; Reviews, 282–288 [review of T. Muir's *The Theory of Determinants in the Historical Order of Development*, vol. 3; etc.].

MATHEMATICS TEACHER, volume 14, no. 2, February, 1921: "Outstanding pedagogical principles now functioning in high-school mathematics" by G. W. Myers, 57–63; "The geometry of the junior high school" by J. C. Brown, 64–70; "Algebraic magic squares" by H. P. McLaughlin, 71–77; "The outlook with regard to school mathematics" by W. P. Webber, 78–84; "Mathe-

matics in Stuyvesant High School" by W. E. Breckenridge, 85–87; "Articulation of junior and senior high school mathematics" by J. K. Van Denberg, 88–94; News and Notes, 95–101; Discussion, 102–103; New Books, 104–105—No. 3, March: "Terms and symbols in elementary mathematics," 107–118 [A preliminary report by the National Committee on Mathematical Requirements]; "The recitation in mathematics" by J. H. Minnick, 119–123; "Certain mathematical ideals of the junior high schools" by D. E. Smith, 124–127; "Geometry detected by Sherlock Holmes" by B. B. Hedges, 128–136; "Remarks on the report of the National Committee on Mathematical Requirements on College Entrance Requirements" by E. R. Hedrick and H. D. Gaylord, 137–142; "The problem of home work papers" by J. R. Overman, 143–146; "Teaching incommensurables" by Vera Sanford, 147–150; Round Table Discussion, 151–155; News and Notes, 156–158; Book Reviews, 159–160.

MESSENGER OF MATHEMATICS, volume 50, no. 5, September, 1920: "On a Diophantine problem" (third paper, continued) by H. Holden, 65–75; "Circular parts: the general case" by W. W. Johnson, 76–80—Nos. 6–7, October–November: "Factorization of N, treated as a bicomposite, special regard being paid to the sum of its digits and to the consequent possible sums of the digits of its twin factors, after casting out the nines" by D. Biddle, 81–95; "A differential equation occurring in the theory of the propagation of waves" by H. Bateman, 95–100; "Summation of q-hypergeometric series" by F. H. Jackson, 101–112—No. 8, December: "On the generating function of the series $\Sigma F(n)q^n$, where $F(n)$ is the number of uneven classes of binary quadratics of determinant $-n$" by L. J. Mordell, 113–128.

NATURE, volume 107, March 31, 1921: "Electrical theory and relativity" by A. R. [review of J. H. Jeans's *The Mathematical Theory of Electricity and Magnetism* (Cambridge, 1920)], 133–134; "Mathematical text-books" by H. B. H. [review of C. Davison's *The Elements of Plane Geometry* (Cambridge, 1920), of W. G. Dunkley's *A Primer of Trigonometry for Engineers* (London, 1920), of S. B. Gates's *Pure Mathematics for Engineers* (2 vols., London, 1920), of P. J. Haler and A. H. Stuart's *A Second Course in Mathematics for Technical Students* (London, 1920), of W. P. Webber's *Elementary Applied Mathematics* (New York and London, 1920), of S. H. Stelfox's *The Laws of Mechanics* (London, 1920) and of J. W. Landon's *Elementary Dynamics* (Cambridge, 1920)], 134–136—April 28: Review of R. D. Carmichael's *The Theory of Relativity* (second edition, New York, 1920), 264; "The concept of 'space' in physics" by H. Jeffries, 267–268—May 5: "Logs and antilogs" a letter by R. T. A. I., 300–301 ["On p. 7 of *Nature* of March 3 a recommendation is mentioned that when taking out the number corresponding to a logarithm a table of antilogs should be used. Assuming the usual seven-figure work, the opposite course should be followed, because the computer can then write down five figures at once and add the remaining two by means of the difference table; no addition or crossing out is required. Thus for the logarithm 0.1234567 the log table gives 1.3287 for 1234269, and 298 in the 327 difference table gives 91, so we write 1328791. *Vice versa*, having 1.328791, what is the logarithm? The antitable gives 12345 at once, whilst the difference 20 gives 67, so that we write 1234567. No figure requires alteration and the work is done with a minimum of mental strain.

"As one who does a great deal of computation, let me state that my order of preference for usual work is Cotsworth's multiplication table (which is better than Crelle's), then the Triumphator or Brunsviga calculating machine, then Shortrede's table, which in one volume gives both logs and antilogs; but special tables can also be usefully employed. Thus Bottomley for all four-figure work is still the best; for multiplying two figures by four, Peters's table; and for two figures by three, Zimmermann's.

"Amongst the indispensable tables should be included Zech's addition and subtraction log table, which is easy to use and accurate. For eight-figure work the best, if not the only, tables are Bauschinger's and Peters's"].

PHILOSOPHICAL MAGAZINE, sixth series, volume 41, April, 1921: "On the supposed weight and ultimate fate of radiation" by O. Lodge, 549–557; "On a method of analysis suitable for the differential equations of mathematical physics" by W. L. Cawley and H. Levy, 584–607; "Motion and hyperdimensions" by F. Tavani, 647–651—May: "The physical significance of the least common multiple" by N. Campbell and E. C. C. Baly, 707–716.

PROCEEDINGS OF THE AMERICAN ACADEMY OF ARTS AND SCIENCES, volume 56, no. 1, February, 1921: "Acoustic impedance and its measurement" by A. E. Kennelly and K. Kurokawa, 1–42 [Bibliography, 20 titles, pp. 38–39; list of symbols employed, pp. 40–42]—No. 7, April: "Anaximander's book, the earliest known geographical treatise" by W. A. Heidel, 239–288.

REVUE SCIENTIFIQUE, volume 59, March 12, 1921: "Les progrès de l'astronomie physique" by H. Deslandres, 97–101 [First paragraph: "L'extension considérable, prodigieuse, des recherches

scientifiques est un des caractères de notre époque. Les hommes de valeur et les établissements spéciaux qui leur sont consacrés sont en nombre toujours croissant, et cette belle progression est surtout frappante en Amérique. Nous avons vu, dans les cinquante dernières années, les découvertes succéder aux découvertes, et dans toutes les sciences. La physique a été favorisée de façon toute particulière; son domaine, déjà de belle étendue, s'est agrandi de terres nouvelles à la fois très riches et très vastes; les phénomènes de radiation et les phénomènes électriques y ont une place prépondérante"].

SCHOOL SCIENCE AND MATHEMATICS, volume 21, no. 5, May, 1921: "Teaching formulæ in the junior high school" by J. A. Nyberg, 409–417; "The Mathematical Association of America" by G. A. Miller, 418–422; "Diophantine analysis applied to the constructibility of regular polygons" by M. O. Tripp, 422–424; "Some observations concerning the history of science" by E. H. Johnson, 450–453; Problems and solutions, 483–488.

SCIENCE, new series, volume 53, April 29, 1921: "Euclid of Alexandria and the bust of Euclid of Megara" by F. Cajori, 414–415 [States that the portrait bust on a certain old Greek coin which is often published as that of Euclid the mathematician, really represents Euclid of Megara, the philosopher, who was formerly often confounded with his greater namesake]—May 27: "A section of the American Association on the History of Science" by L. C. Karpinski, 500–501 [Last paragraph: "History of science, using science with the inclusive meaning as in the title A. A. A. S., is surely the proper name for the new section now under way"]—June 10: "Inaugural address" by E. F. Nichols, 523–527 [as president of Massachusetts Institute of Technology].

SCIENCE PROGRESS, volume 15, no. 4, April, 1921: "Recent advances in pure mathematics" by Dorothy Wrinch, 517–522 [contains an elementary explanation of nomography, apropos of S. Brodetsky's book on the subject (*1921*, 131–132)]; "DeMoivre's theorem" by R. Ross, 627–628 [First paragraph: "I will be very much obliged to any of our mathematical readers who will be so kind as to inform me where I can find any record of the following proposition—which shows that DeMoivre's famous theorem connected with complex numbers is only a particular case of the iteration—that is, the *operative* involution—of a real algebraic function of which one of the parameters is reduced to zero. I have known the proposition for many years, and indeed indicated it in my paper on 'Operative Involution' in *Science Progress*, No. 50, p. 288, October, 1918, in some examples at the end, and in No. 51, p. 486, January, 1919, last example; but I have searched in vain for it through my books—even in the works of Hamilton, Tait, and Joly on quaternions, on which subject it has an important bearing"]; "Highways and byways in the theory of numbers" by L. J. Mordell, 647–652 [Review of L. E. Dickson's *The History of the Theory of Numbers*, volume 2 (Washington, 1920). First sentences: "All mathematicians interested in the theory of numbers, and this means sooner or later most pure mathematicians, will welcome volume ii of Prof. Dickson's 'Chronological History.' It notes practically everything written on the subject, sums up the results of a paper in a few lines, and might serve as a model of orderly arrangement. This history adds considerably to the increasing debt of mathematicians to America, and is a real necessity in their libraries"].

SCIENTIFIC MONTHLY, volume 12, May, 1921: "The history of mathematics" by E. W. Brown, 385–413 [Lecture delivered at Yale University, February 26, 1920. First paragraph: "The earliest dawn of science is without doubt not different from that of intelligence. But the civilized man of to-day, far removed as he is from the lowest of existing human races, is probably as far again from the being whom one would not differentiate from the animals as far as mental powers are concerned. What this difference is, neither ethnologist nor psychologist can yet tell. Perhaps the nearest approach to a definition, at least from the point of view of this article, is contained in the distinction between unconscious and conscious observation. We are familiar with both sides even in ourselves; records can be impressed on the brain and remain there apparently dormant until some stimulus brings them to fruition, and again, record and stimulus can appear together so that a train of thought is immediately started"]; "The history of science as an error breeder" by G. A. Miller, 439–443.

TECHNOLOGY REVIEW, volume 23, January, 1921: "Professor Cecil Hobart Peabody" by W. Hovgaard, 12–14 + portrait.

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 22, April, 1921: "On division algebras" by J. H. M. Wedderburn, 129–135; "Oscillation theorems for the real, self-adjoint linear system of the second order" by H. J. Ettlinger, 136–143; "New proofs of certain finiteness theorems in the theory of modular covariants" by Olive C. Hazlett, 144–157; "On the convergence of certain trigonometric and polynomial approximations" by D. Jackson, 158–166; "Determination of all general homogeneous polynomials expressible as determinants with linear

elements" by L. E. Dickson, 167–179; "Pseudocanonical forms and invariants of systems of partial differential equations" by A. L. Nelson, 180–197; "Arithmetical paraphrases (II)" by E. T. Bell, 198–219; "On the zeros of solutions of homogeneous linear differential equations" by C. N. Reynolds, Jr., 220–229; "A generalization of the Fourier cosine series" by J. L. Walsh, 230–239; "Polynomials and their residue systems" (to be continued) by A. J. Kempner, 240–266.

UNIVERSITY BULLETIN, Louisiana State University, new series, volume 13, no. 2, February, 1921: *Fundamental Aspects of Mathematical Training* by S. T. Sanders, 30 pages [First paragraph: "The tendency is outstanding in secondary education to exclude from courses of study the subject that does not have 'value in relation to other topics and to time involved.'[1] 'The ideal of practicality has now entered the schools with telling force. It has been manifested in its demand for vocational training, and it is reconstructing the older cultural training by eliminations and additions. Materials once accepted without question when schools had a margin of energy are now displaced by the pressure of new demands.'[2] 'But the meaning of the practical is not that of the eighteenth century. Arithmetic now represents tools which the child needs to control his present and potential quantity situations'[3]"].

UNTERRICHTSBLÄTTER FÜR MATHEMATIK UND NATURWISSENSCHAFTEN, volume 27, February 26, 1921: "Die Bedeutung der nichteuklidischen Geometrie für den Elementar-Unterricht" by A. Schülke, 3–6; "Rechentafel zur Auflösung zweier Gleichungen ersten Grades mit zwei Unbekannten" by P. Luckey, 8–9; "Das grösste einem gegebenen Kreisabschnitte eingeschriebene Rechteck und die grössten einem gegebenen Kugelabschnitte einbeschriebenen Kreiszylinder," by —. Friese, 9–11; "Zur Konstruktion des grössten Rechtecks in einem Kreissegment" by W. Gaedecke, 11–12; "Grenzwerte symmetrischer Verbindungen der Winkelfunktionen am Dreieck" by A. Emmerich, 12.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 52, nos. 1–2, February 17, 1921: "Ein Weg zur Relativität für die Schule" by A. Schoenflies, 1–13; "Zur Theorie der komplexen Zahlen" by W. Schwan, 13–17; "Graphische Behandlung der Zinsrechnung" by A. Schülke, 17–19; "Für und wider das abgekürzte Rechnen" by K. Becker, 19–24; "Zur Reform des mathematischen Hochschulunterrichts" by E. Kamke, 24–26; "Die Möbiussche Form des Brechungsgesetzes" by R. Böger, 27–31; "Eine Bemerkung zum d'Hondtschen Wahlverfahren" by —. Behmann, 32–34; "Der Lehrsatz des Pythagoras als Sonderfall eines Höhensatzes" by A. Maennersdoerfer, 35–36; "Die Winkelmessung des Babyloniers, des Artilleristen, und des Mathematikers" by P. Luckey, 36–37; "Aufgabenrepertorium," 38–41; "Ueber den Nichtgebrauch und den Missbrauch der Mathematik bei den Begabten-prüfungen" by M. Vaërting, 41–46; "Bücherbesprechungen," 46–55.

UNDERGRADUATE MATHEMATICS CLUBS

All reports of club activities should be sent to E. L. Dodd, 3012 West Ave., Austin, Texas

CLUB ACTIVITIES.

THE MATHEMATICS CLUB OF HUNTER COLLEGE, New York City.

[*1918*, 187.]

Prior to October, 1919, meetings were held monthly, as follows:

January, 1919: "Sun dials" by Anita Rosenthal '19.

March: Reception to freshmen.

April: "Prominent mathematicians of the present day" by Edith Gaddis '21; "Paper folding," under the direction of Anita Rosenthal '19.

May: "Perfect, amicable, square, and triangular numbers" by Laura Guggenbuhl '22.

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[1] "Bulletin, 1920, No. 1, U. S. Bureau of Education."

[2] "'High School Mathematics,' by G. W. Evans, Headmaster of Charleston High School."

[3] "Bulletin, 1917, No. 10, U. S. Bureau of Education."

In October, 1919, a reorganization of the club was effected with a view to placing all responsibility for the success of the club upon the student members. In November the following officers were elected: President, Melanie Rohrer '20; vice-president, Helen Myer '20; secretary, Helen Deaken '20; treasurer, Charlotte Warnock '21, publicity manager, Edith De Hondt '20, faculty adviser, Marcia Latham, instructor.

December: Catch problems, brought by volunteers and discussed by the club. Members were invited to drop further solutions in the Mathematics Club mail box.

Since the reorganization, the arrangement is to have three afternoon meetings each semester, the first one being purely social. In addition, two regular meetings a month are held during the noon recess hour. Outside speakers are invited each semester. Recently a "Problem Chapter" has been formed with Laura Guggenbuhl '22 in charge for the discussion and solution of problems too technical for the regular meetings. Frequent reference is made to the problems published in the MONTHLY. The following are the topics of meetings held during 1920.

January 10, 1920: "Scales of notation" by Laura Guggenbuhl '22 and Helen Myer '20.

February 23: "Magic squares" by Eva Thornton '22.

March 8: "Card tricks" by Mary Bailey '21.

March 23: "The fourth dimension" by Anna Colligan '21, Jessie McCormick '22, and Sarah Block '22.

April 5: Book reviews by Monica Gilloran '21, Edna Dramer '23, Charlotte Warnock '21, and others.

May 17: Election of officers: President, Catherine Cronin '21; vice-president, Mary Bailey '21; secretary, Dora Aronoff '21; treasurer, May Warnock '22; publicity manager, Sarah Karnis '22; faculty adviser, Marcia Latham, instructor.

October 4: "Geometrical fallacies" by Margaret Cronin '21.

November 1: "The use of the slide rule in the business world" by Sarah Mones '21.

November 15: "Business calculations in the middle ages" by Laura Guggenbuhl '22, Jessie McCormick '22, and Sarah Block '22, with lantern slides shown by Professor Lao Simons.

December 6: A demonstration of calculating machines by Mr. O'Brien, of the Dalton Adding Machine Company.

December 24: Christmas Party.

A red letter day during 1920 was a special afternoon meeting addressed by Professor David E. Smith on the teaching of mathematics in the junior high schools.

The average attendance at the noon meetings has been forty and that at the afternoon social meetings one hundred. A special effort has been made to make the social meetings popular and at the same time keep the flavor of a mathematics club. The following are some special features of different meetings: Miss Polly

Gon, the magician, who divined ages, numbers on a blackboard visible to the audience but concealed from her, throws of dice, and so forth; at another meeting the chief entertainment was a contest based on the following puzzles devised by Beatrice Meyer '22. Each of these is a mathematical term when translated: 1. C + to possess something = ————; 2. To make an attempt + to fish = ————; 3. A parrot + part of the verb *to go* = ————; etc.; students sometimes furnish original songs, which together with more familiar ones, such as "Euclid had a little book" and "Every little symbol has a meaning all its own" are sung at the close of the meetings. The club has a special bulletin board for announcements, clippings, special problems, etc.

(Report by Miss Aronoff.)

THE WHITE MATHEMATICS CLUB AT THE UNIVERSITY OF KENTUCKY, Lexington, Ky.

[*1918*, 90, 451; *1919*, 309.]

The following are programs of recent meetings of the Club.

May 7, 1919: "Monge's solution of the non-integrable complete differential equation" by Professor J. M. Davis.

May 12: "Class and one-to-one correspondence" by Frank Tuttle '20.

May 20: "Review of nilpotent algebras generated by two units, i and j, such that i^2 is not an independent unit" by Dr. G. W. Smith, instructor.

May 26: "Some linkages considered analytically" by Harvey Pettitt Gr.

June 2: "Applications of vector analysis to the theory of helices" by William Elliott Gr.

September 30, 1919: Election of officers: President, Professor Davis; secretary, Dr. Smith.

October 7: Report of mathematics meetings held at Ann Arbor, Michigan, September 2–5, 1919, by Dr. Smith.

October 14: "Quadratic transformations" by Professor P. P. Boyd.

October 21: "Discontinuities" by William Elliott Gr.

October 28: "Tests of mathematical ability" by Professor Davis; "The Committee on Mathematical Requirements" by Professor Boyd; "The coefficient of correlation" by Dr. Smith.

November 4: "To draw a circle tangent to three given circles by means of inversion" by W. E. Armentrout Gr.

November 12: "Determination of the angular velocity from the accelerations of non-collinear points of a rigid body" by Professor E. L. Rees.

November 18: "Some angles of the right triangle" by Frances Kimbrough '20.

November 25: "Singular case of conics" by Professor H. H. Downing.

December 2: "Paper folding" by Frank Tuttle '20.

December 9: "Introduction to the theory of statistics" by Dr. Smith.

December 16: "Bolshevism in mathematics" by Professor Rees.

February 17, 1920: "Wireless telegraphy" by Ernest Baulch '21.

February 24: "Quadratic transformations" by Professor Boyd.

March 2: "Magnus's transformations" by Jesse Osborn Gr.
March 9: "Some peculiar functions" by Dr. Smith.
March 16: "Some propositions about circles proved by means of properties of the radical axes" by W. E. Armentrout Gr.
March 23: "Taylor and Taylor's series" by Professor Downing.
March 30: "Some properties of integral numbers" by Frances Kimbrough '20.
April 6: "Quadratic transformations, Steiner's method" by Jesse Osborn Gr.
April 13: "Discussion of the preliminary report of the Committee on Mathematical Requirements."
April 27: "Bohr's picture of the atom" by Professor W. B. Angel, physics department.
May 11: "Cubic and quartic curves with a line of symmetry" by Professor Rees.
May 20: "The mathematical theory of investment" by Dr. Smith.
October 20, 1920: "A geometric proof of a theorem concerning the roots of a quartic" by Professor Rees. Election of officers: President, Professor Rees; secretary, Dr. Flora E. Le Stourgeon, instructor.
October 26: "Derivatives from the expansion of the function" by Professor Downing.
November 2: "Conversion of series into continued fractions" by W. E. Armentrout Gr.
November 11: "Functions of lines" by Dr. LeStourgeon.
November 16: "Curve tracing" by L. P. Rippy Gr.
November 23: "Applications of mathematics to economics" by W. E. Payne, instructor.
November 30: "The trisection of an angle and the duplication of a cube" by D. C. Duncan '22.
December 7: "Differential equations (Review of Bateman's *Differential Equations*)" by Professor Davis.
December 14: "Theory of the slide rule" by Nelson Conkwright '22.
January 13, 1921: "The Hessian, the Steinerian, the Cayleyan" by Professor Boyd.
January 18: "Curve tracing" by Professor Rees.
February 14: "Some properties of the logarithmic functions" by Professor Downing.
February 23: "Some ancient methods of computation" by Professor Boyd; "Two mathematical paradoxes" by Professor Downing; "Mathematics and religion" by Professor Davis.
March 2: "Stieltjes's integrals and linear functionals" by Dr. LeStourgeon.

(Report by Dr. LeStourgeon.)

PI MU EPSILON FRATERNITY (INC.), Syracuse University, Syracuse, N. Y.
[*1918*, 271.]

Officers for the year 1919–20 were as follows: Director, Mrs. Mary Harwood, instructor; vice-director, J. J. Hopfield, instructor in physics; secretary, Mary

Hutchinson '20; treasurer, Edward Houghtaling '21; librarian, Marion Jarvis '20.

During the college year six meetings were held, a Christmas party, and a picnic. One graduate student and twelve undergraduates were elected members. A chapter at Ohio State University was established. The programs were as follows:

November 10, 1919: "Travels through France" by Professor W. H. Metzler; "Graduate work at the University of Edinburgh" by Dr. J. J. Nassau.

December 1: "The region at infinity" by Dr. T. S. Yang.

February 23, 1920: "Some new equations in graduations" by Dr. Nassau.

April 19: "The theory of relativity" by Dr. Nassau.

May 10: "On some special curves" by Edna Lawrence '20.

(Report by Dr. Nassau.)

THE PASCAL CIRCLE OF TRINITY COLLEGE, Washington, D. C.

[*1920*, 425.]

The officers for the year 1920–21 are as follows: President, Margaret Walsh '21; vice-president, Elizabeth Herbert '22; secretary and treasurer, Margaret Kelly '23; faculty adviser and honorary president, Professor Marie C. Mangold.

October: The name, purpose, and requisites for membership of the society were explained by the president. Recreational problems in geometry were presented.

November: "The value of mathematics to the physiologist and physician" [1] by Elizabeth Herbert '22. At this meeting it was decided to make the discussions of the year hinge on the relation of mathematics to the various other sciences, and to have three members appointed at each meeting to bring in recreational problems for the next meeting.

December: "Why automobiles skid" by Frances McFadden '23; "The relation of Mathematics to English" by Lillian Manganero '22. This refuted the frequent charge that skill in one excludes skill in the other.

(Report by Miss Walsh.)

THE JUNIOR MATHEMATICAL CLUB, University of Wisconsin, Madison, Wis.

[*1918*, 188, 457; *1920*, 224.]

The present officers are as follows: President, Phillip Dowling '21; vice-president, Gretchen Votteler '21; secretary and treasurer, Alice N. Tucker '22.

The programs for the past year are as follows:

January 22, 1920: "Integers" by Professor E. B. Skinner.

February 26: "Graphical calculations" by Professor Arnold Dresden.

March 11: "Mathematics in physics" by Harold Laird '22 and Phillip Dowling '21.

March 25: "What is a wave?" by Professor C. S. Schlichter.

[1] A paper by Professor H. B. Williams, with a title similar to this, was delivered before the Mathematical Association of America (*1920*, 95–97) in December, 1919. The main part of the paper was published in *Mathematics Teacher*, March, 1920.—EDITOR.

April 8: "Geometric probabilities" by Dr. Florence Allen, instructor.
April 22: "Squaring the circle" by Professor W. W. Hart.
May 27: Election of officers; annual reports.
October 28: "The semi-regular solids of Archimedes" by Professor E. B. Van Vleck.
November 11: "Mathematics and logic" by Professor Dresden.
December 2: "Methods of mathematics" by Victor Von Szeliski.
December 16: "Wave motion" by Professor Schlichter.
January 13, 1921: "Expansion of fundamental laws of algebra" by Professor Skinner.
January 27: "The value of π" by Grace Desimval '21.
February 24: "Interesting applications of the theory of probability" by Professor Max Mason.

(Report by Miss Tucker.)

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND H. P. MANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. Problems containing results believed to be new, or extensions of old results are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible, however, the editors will be glad to assist members of the Association with their difficulties in the solution of such problems.]

2920. Proposed by N. P. PANDYA, Sojitra, India.

Construct a triangle, having given the base, the angle between the base and the median on it, and the difference of the remaining two sides.

2921. Proposed by J. W. CLAWSON, Ursinus College, Pa.

ABC is a triangle cut by a transversal PQR, so that A, P; B, Q; and C, R are opposite vertices of a complete quadrilateral. Draw CD, PF, QE, chords in the circles circumscribing, respectively, triangles ABC, BRP, AQR, all these chords being parallel to AB.

Prove that (i) D, E, F are collinear; (ii) the line DEF passes through the Wallace point of the quadrilateral (the point of concurrency of the circles mentioned above); (iii) the line DEF intersects AB at the point of tangency to AB of the parabola which touches the four sides of the quadrilateral.

2922. Proposed by the late A. M. KENYON.

A telephone engineer desires the general solution of the following differential equation:

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}=\frac{a}{\sqrt{y}}.$$

2923. Proposed by C. N. SCHMALL, New York City.

The corner of a page of a book is turned down in such a manner that the triangle formed has a constant area. Show that the locus of the corner is an oval of the curve,

$$r^2 = a^2 \sin 2\theta.$$

2924. Proposed by FLORENCE P. LEWIS, Goucher College.

Given a triangle and a conic. Through each vertex of the triangle there pass two lines harmonic to the tangents through that point and to the sides of the triangle. Prove that the six lines so found pass by threes through four points.

2925. Proposed by F. V. MORLEY, New College, Oxford, Eng.

A regular polygon of $2n+1$ sides will have only $n-1$ diagonals of different lengths (*e.g.*, the regular heptagon has two distinct diagonals). Call the side of such a polygon a_1, and the $n-1$ diagonals in order of size $a_2 \cdots a_n$. Then if the circumscribed circle has radius unity, $\sum_{i=1}^{n} a_i^2 = 2n+1$; in words, the sum of the squares of the distinct lengths obtained by joining an odd number of regularly spaced points on a unit circle is equal to the number of such points.

2926. Proposed by T. M. SIMPSON, Randolph-Macon College, Ashland, Va.

Solve the differential equation,

$$(y + x^2)dx + (x - x^2y)dy = 0.$$

2927. Proposed by PHILIP FRANKLIN, Princeton University.

Prove that the only positive integral values greater than unity which satisfy the equation $3^x - 2^y = \pm 1$ are $x = 2$, $y = 3$. (Cf. Carmichael, *Diophantine Analysis*, 1915, p. 116, exercise 69.)

NOTES.

22. Huge Numbers.—"What is the largest number that we can express by three digits?" The answer is $N = 9^{(9^9)}$, that is $9^{387,420,489}$. C. A. Laisant drew attention to this number in his *Initiation Mathématique*, Paris, 1906 (English edition, London, 1913). He there remarks that in decimal numeration this number would have 369,693,100 figures. To write it on a single strip of paper, supposing that each figure occupied a space of one fifth of an inch, the length of the strip would need to be 1,166 miles, 1,690 yards, 1 foot, 8 inches. In this connection C. E. Guillaume remarked (*Revue Générale des Sciences*, vol. 17, 1906, p. 878) that under the same conditions to write $10^{(10^{10})}$, we would need a strip of paper long enough to encircle the earth.

Writing in May, 1913, A. C. D. Crommelin stated (*Journal of the British Astronomical Association*, vol. 23, pp. 380–381) that he had come across the problem with which this note commences "in an old logarithm book." By the aid of 61-figure logarithms of certain numbers given in Hutton's tables Dr. Crommelin found $\log N = 369{,}693{,}099.6315703587\cdots$; whence the number of figures indicated above. He found the first 28 of the figures to be 428,124,773,175,747,048,036,987,115,9 and the last three to be 289.

"A knowledge of 30 figures out of 300 million," he continued, "may seem trifling, but in reality the error involved in taking all the remaining figures as zeros is only one part in a thousand quadrillions. If the number were printed with 16 figures to an inch (about the tightest packing for decent legibility), it would extend over 364.7 miles. . . . If printed in a series of large volumes we might get 14,000 figures to a page, and with 800 pages to the volume it would fill 33 volumes. There are more than twice as many digits in the number as there are letters in the whole of the *Encyclopædia Britannica*.

"To find the largest number suggested by sideral astronomy I took the following. Both Very and See have expressed the opinion that certain visible objects may be at a distance of a million light years; I imagine a solid sphere of platinum of this radius, and find how many electrons it contains. From Duncan's *The New Knowledge*, p. 65, I find that the log of the number of electrons in a cubic centimetre of water is 16.469. Taking the density of platinum as 21.5 the log of the number of electrons in a cubic inch of it is 19.016, and the log of the volume of the

huge sphere in cubic inches is 71.3362. Whence the log of the number of electrons it contains is 90.352, and the corresponding number is 225 followed by 88 zeros. At 16 figures to the inch this would take $5\frac{3}{4}$ inches. . . .

"To find the radius of a sphere of platinum that would contain 9^{99} electrons, we must multiply our million-light-year radius by a number whose log is 123,231,003.093, *i.e.*, the multiplier is 1239 followed by 123,231,000 zeros. In fact, that gigantic sphere would exceed the million-light-year sphere in a far higher ratio than that exceeds the size of one electron.[1] Hence we may take it as morally certain that we can write with three digits a number vastly exceeding the number of electrons in the whole of creation, which is a somewhat startling fact. Indeed, even the number 4^{4^4} (which is 13407813 followed by 147 other figures) probably exceeds the number of electrons in creation. At least it equals the number of electrons in a solid platinum sphere that exceeds the million-light-year sphere in the same proportion that that exceeds a sphere 206 inches in radius."

In May, 1915, D. G. McIntyre considered (*Journal of the British Astronomical Association*, vol. 26, pp. 46–47) the "rather pretty" problem of determining the last figures of N, three of which, 289, were given by Dr. Crommelin. He found the last eight figures to be 17177289.

In *The Observatory* for July, 1920, H. H. Turner refers to N and to his choice, as rather simpler for expression and not very different in magnitude, of the number $10^{(10^{10})}$ which, when written out fully, is unity followed by ten thousand million zeros. "The validity of the substitution remained unchallenged until the other day, when Dr. Crommelin realised that injustice had been done. He accused me of treating as comparable, let us say, a billion times the distance of Dr. Shapley's furthest star cluster and a wave-length of light; for the ratio of these would, he said, be no greater than that of $10^{10^{10}}$ to 9^{9^9}. The accusation could not be repelled, and I tendered apologies as gracefully as the magnitude of the error would allow. But the accuser had apparently not reaped his advantage to the full. A few days later I received the following post card from him:

"I greatly understated the ratio of $10^{10^{10}}$ to 9^{9^9}, which is a number of some 900 million figures, and would be several thousands of miles long if written out at 16 figures to the inch. On the other hand, the ratio I mentioned yesterday (viz., a billion times the distance of Shapley's furthest cluster to the billionth of an inch) would have something under 50 figures in it, and could be written in the width of this postcard.

"I fear the framing of a suitable acknowledgment is beyond me."

Finally, in *Journal of the British Astronomical Association*, April, 1921, J. W. Meares comments on $9!^{(9!^{9!})}$ and finds that the value of his number is greater than 10 to the power $10^{2000000}$ but less than 10 to the power $10^{2000001}$. On this Dr. Crommelin commented: "If one allows the introduction of algebraic symbols the number $\infty^{\infty^{\infty}}$ has some claims on our attention. Perhaps I may be allowed to quote an old college rhyme:

"'There was a professor of Trinity
Who found the square root of infinity;
But in counting the digits
He was seized with the fidgets,
Dropped Science and took to Divinity.'"

One wonders if reference is here made to George Salmon, of Trinity College, Dublin, whose classic mathematical works have throughout the world been the delight of generations, but whose works on "Divinity," written after he had "dropped" mathematics, are known to few! ARC.

[1] "Indeed we should have to carry out enlargement in the ratio of a million-light-year sphere to an electron more than a million times in succession before we get a sphere of the size enlarged."

23. The Wallace Line and Wallace Point, and Some Generalizations.—In 1799, William Wallace[1] stated the theorem: Given a circle and a triangle inscribed in it, the feet of the perpendiculars on the sides of the triangle from any point of the circle are collinear.[2] Such a line is the Wallace line of the point. The first generalization of the theorem was in 1822 by Poncelet[3] who showed that the perpendiculars on the sides of the triangle may be replaced by obliques making, in cyclic order, equal angles with the sides. Both Wallace and Poncelet seem to have arrived at their results[4] by considering properties of the parabola: (*a*) the circumscribed circle of a triangle tangent to a parabola passes through its focus;[5] (*b*) the feet of the perpendiculars from a focus of a parabola on its tangents lie on the tangent at the vertex of the parabola.

As an application of his theorem, Wallace considered the question[6] of describing a parabola tangent to four given straight lines; he remarks that the focus of the parabola is determined by finding the second point of intersection of the circles circumscribing any two of the four triangles formed by the four given lines. In other words the four circles meet in a point—the Wallace point.[7] Without reference to the parabola, Wallace formulated the theorem again in 1804. It appeared also in Bland's *Geometrical Problems*, Cambridge, 1819, p. 259, before

[1] *The Mathematical Repository*, March, 1799, p. 111.

[2] This same theorem was given by Servois in February, 1814 (*Annales de Mathématiques* (Gergonne), vol. 4, p. 251). In a footnote he states the theorem in another form: "Through a point on the circumference of a circle three chords are drawn. Circles are described on these chords as diameters. The second points of intersection of these circles, taken in pairs, are collinear."

[3] *Traité des Propriétés Projectives*, 1822, p. 270.

[4] *The Mathematical Repository*, September, 1798, p. 81; Poncelet, *l.c.*, pp. 218–219.

[5] Lambert, *Insigniores Orbitae Cometorum Proprietates*, 1761, p. 5.

[6] *The Mathematical Repository*, March, 1799, p. 81.

[7] The centers of the four circles lie on a circle through the Wallace point—a result stated without proof by Steiner in 1828, *Annales de Mathématiques* (Gergonne), vol. 18, p. 302. From the above it is clear that there is scant justification for calling the Wallace point, the Miquel point, as has been done by S. Kantor (*Comptes rendus de l'Académie des Sciences de Vienne*, 1878), since Miquel derived the result in the periodical *Le Géomètre* (founded by Guillard), 1836, and later in Liouville's *Journal*, vol. 3, 1838, p. 486. So also J. L. Coolidge refers (*A Treatise on the Circle and the Sphere*, 1916, p. 87) to "the Miquel point of the four lines."

In 1871 W. K. Clifford remarked [in his *Common Sense of the Exact Sciences*, 1885, pp. 80–81] that Wallace's theorem concerning concurrent circles is the third of a series:

"If we take any two straight lines they determine a point, viz., their point of intersection.

"If we take three straight lines we get three such points of intersection; and these three determine a circle, viz., the circle circumscribing the triangle formed by the three lines.

"Four straight lines determine four sets of three lines by leaving out each in turn; and the four circles belonging to these sets of three meet in a point.

"In the same way five lines determine five sets of four, and each of these sets of four gives rise, by the proposition just proved, to a point. It has been shown by Miquel [*l.c.*], that these five points lie on the same circle.

"And this series of theorems has been shown [W. K. Clifford, 'Synthetic proof of Miquel's theorem,' *Oxford, Cambridge and Dublin Messenger of Mathematics*, vol. 5, 1871, p. 124] to be endless. Six straight lines determine six sets of five by leaving them out one by one. Each set of five has, by Miquel's theorem, a circle belonging to it. These six circles meet in the same point, and so on forever. Any even number ($2n$) of straight lines determines a point as the intersection of the same number of circles. If we take one line more, this odd number ($2n + 1$) determines as many sets of $2n$ lines, and to each of these sets belongs a point; these $2n + 1$ points lie on a circle."

it was given by Steiner in 1828 (compare problem 2898 of this MONTHLY, *1921*, 228).

Professor C. N. Mills of Tiffin, Ohio, suggested the following problem: "Given any triangle cut by a transversal through a fixed point on the base produced; through the fixed point and the intersections of the transversal with each side and the adjacent vertex at the base circles are drawn. Show that the locus of the intersection of these two circles is the circumcircle of the given triangle." From what has been given above it is clear that the point in which the circles meet is the Wallace point for the particular position of the transversal, which is the Poncelet line for that Wallace point.

Another generalization of Wallace's theorem stated without proof[1] (anonymously) in July, 1823, was as follows: (*a*) If from any point of a circle concentric with the circumscribed circle of a triangle, perpendiculars are dropped on the three sides, the area of the triangle, whose vertices are the feet of the perpendiculars, is constant. When this circle becomes the circumscribed circle the area vanishes. (*b*) If two circles concentric with the circumscribed circle are such that the sum of the squares of their radii is equal to twice the square of the radius of the circumscribed circle the two triangles formed as above are equivalent.

In 1870 Combette gave among other results:[2] (*a*) If P be an assumed point, and D, E, F its projections on the triangle ABC, the locus of P, when the area of the triangle DEF is constant, is the circumference of a circle concentric with the circumscribed circle of ABC; (*b*) For every value of the area of DEF included between zero and one fourth of ABC, the locus of P will consist of two circumferences concentric with the circumscribed circle, the one interior and the other exterior to it; (*c*) The sums of the squares of the radii of these two circumferences will be double the square of the circumscribed radius; (*d*) The locus of P does not change its nature or its center when the perpendiculars let fall on the sides become lines all making equal angles with the sides; (*e*) If the triangle ABC is replaced by a plane polygon, and the projections on its sides of a point P in the plane are joined in order, the locus of P when this area is constant is still a circle which has always the same center whatever be the value of the area;[3] (*f*) When the point P is taken in space and projected on the sides of a plane polygon, the locus of P, when the area obtained by joining the projections is constant, becomes a cylindrical surface of revolution whose axis, perpendicular to the plane of the polygon, is always the same for all values of the area.

The locus of the points whose projections on the planes of the faces of a

[1] *Annales de Mathématiques* (Gergonne), vol. 14, p. 28; Gergonne was probably the author. Analytic proofs are given on pages 280–293, the latter being by the great Sturm.

[2] *Revue des Sociétés Savantes*, vol. 5, pp. 203–233; compare J. S. Mackay, *Proceedings of the Edinburgh Mathematical Society*, vol. 9, pp. 86–87.

[3] This result was first enunciated for a *regular* polygon, without proof, by L'Huilier in *Bibliothèque Universelle*, March, 1824, p. 169. Proofs were given in *Annales de Mathématiques* (Gergonne), vol. 15, by "abonné," July, 1824, pp. 45–55; and by Sturm, February, 1825, pp. 250–252. This was generalized as above (and with further interesting results) by Steiner, in *Journal für die reine und angewandte Mathematik*, vol. 1, 1826, pp. 51–52; see also vol. 2, 1827, p. 265, and a further generalization (p. 263).

tetrahedron are coplanar is a cubic surface, S, through the edges of the tetrahedron and having the vertices of the tetrahedron as nodes.[1]—The surfaces S, for the 15 tetrahedra determined by any six planes, meet in a point P (corresponding to the Wallace point above), and the pedal planes of P for the 15 tetrahedra are coincident.—Steiner remarked, in 1845, that S is the locus of the centers of the hyperboloids for which, when their equations are in standard form,

$$(1/a^2) + (1/b^2) = 1/c^2,$$

and for which a given tetrahedron is self-polar.

Among many discussions of the surfaces, S, the following may be mentioned: by Geiser, in *Journal für die reine und angewandte Mathematik*, vol. 69, 1868, p. 199 f.; by F. E. Eckhardt, in *Mathematische Annalen*, vol. 5, 1872, pp. 30–49; by E. Janhke, *Archiv der Mathematik* (Grunert), third series, vol. 4, 1903, especially pp. 275–276; by J. Neuberg, *Archiv der Mathematik* (Grunert), third series, vol. 16, 1910, p. 18 f.; and by W. H. Salmon, in *Archiv der Mathematik* (Grunert), third series, vol. 18, 1911, pp. 154–164. ARC.

SOLUTIONS.

2719 [1918, 302]. Proposed by R. P. BAKER, University of Iowa.

Show that[2] $2x(\log x)^2 - x(x-1)(x+3)\log x + (x-1)^2(3x-1)$ is negative for $1 < x < \infty$.

SOLUTION BY OTTO DUNKEL, Washington University.

Denoting the given expression by $f(x)$, its first two derivatives may be written as follows:

$$f'(x) = 2(\log x)^2 - (x-1)(3x+7)\log x + 8(x-1)^2,$$

$$f''(x) = \frac{3x^2+2x-2}{x}\varphi(x), \qquad \varphi(x) = \frac{(13x-7)(x-1)}{3x^2+2x-2} - 2\log x.$$

It will be observed that $f(1) = f'(1) = f''(1) = 0$. It will be shown that $f''(x)$ is negative for $x > 1$ and it will then follow that $f'(x)$ is also negative, and hence $f(x)$ is likewise negative for $x > 1$. The derivative of $\varphi(x)$ may be written

$$\varphi'(x) = -\frac{2(x-1)^3(9x-4)}{x(3x^2+2x-2)^2}$$

and it is clearly negative for $x > 1$. Since $\varphi(1) = 0$, it follows that $\varphi(x) < 0$. The first factor of $f''(x)$, which may be denoted by $\psi(x)$, has the roots $-(\sqrt{7}+1)/3$ and $x_1 = (\sqrt{7}-1)/3 = .549$, and hence $\psi(x) > 0$ for $x > 1$. Therefore, $f''(x) < 0$ for $x > 1$ and the desired result is proved.

The form of $\varphi'(x)$ shows that the first of the derivatives of $f(x)$ which do not vanish for $x = 1$ is $f^{\text{vi}}(x)$. The expressions above give at once $f^{\text{vi}}(1) = -20$ and this shows that $f(x)$ is also negative for values of $x < 1$. It will now be shown that the interval for such values extends down to $x = 0$. In the interval $0 \leqq x < 1$ the derivative $\varphi'(x)$ vanishes only for 4/9 and it is negative before and positive after this value. Hence, $\varphi(x)$ has a minimum at this point. It will be found that $\varphi(4/9) = .312$, and so $\varphi(x)$ is positive from 0 to x_1. It is negative from x_1 to 1, since it vanishes at $x = 1$ and $\varphi'(x)$ is positive at all other points of this interval. On the other hand $\psi(x)$ is negative from 0 to x_1 and positive from x_1 to 1. At x_1 the product $\psi(x)\varphi(x)$ has a finite negative value and hence $f''(x)$ is negative from 0 to 1. From this follows that $f'(x)$ is positive and $f(x)$ is negative in the interval $0 \leqq x < 1$. Thus $f(x)$ is negative at every point of the interval $0 \leqq x < \infty$ except at the point $x = 1$ where it vanishes.

[1] In *Annales de Mathématiques* (Gergonne), April, 1814, vol. 4, p. 320, the following problem was proposed (by Gergonne?) for solution: "The feet of the perpendiculars dropped on the faces of a tetrahedron from a point on the circumscribed sphere are coplanar." The incorrectness of this statement was proved by Durrande, in the same periodical, February, 1817, vol. 7, p. 255.

[2] It should be noted that this relation holds for "Naperian" but *not* for common logarithms. —EDITORS.

2812 [1920, 81]. Proposed by C. N. SCHMALL, New York City.

If $F(x, y, z)$ be a homogeneous function of x, y, z which becomes $\phi(u, v, w)$ by elimination of x, y, z by means of the equations, $\partial F/\partial x = u$, $\partial F/\partial y = v$, $\partial F/\partial z = w$; show that

$$\frac{\partial F}{\partial u}\Big/x = \frac{\partial F}{\partial v}\Big/y = \frac{\partial F}{\partial w}\Big/z.$$

SOLUTION BY THE PROPOSER.

The result is incorrectly stated. It should read,

$$\frac{\partial \phi}{\partial u}\Big/x = \frac{\partial \phi}{\partial v}\Big/y = \frac{\partial \phi}{\partial w}\Big/z.$$

Let k be the degree of the homogeneous function F. Then by Euler's theorem on homogeneous functions,

$$xu + yv + zw = kF. \tag{1}$$

By partial differentiation of (1) with respect to x, we obtain

$$u + x\frac{\partial u}{\partial x} + y\frac{\partial v}{dx} + z\frac{\partial w}{\partial x} = ku,$$

or

$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} = (k-1)u. \tag{2}$$

Similarly, we get

$$x\frac{\partial u}{\partial y} + y\frac{\partial v}{\partial y} + z\frac{\partial w}{\partial y} = (k-1)v \tag{3}$$

and

$$x\frac{\partial u}{\partial z} + y\frac{\partial v}{\partial z} + z\frac{\partial w}{\partial z} = (k-1)w. \tag{4}$$

Again,

$$u = \frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial x} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial x}, \tag{5}$$

$$v = \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial y} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial y}, \tag{6}$$

$$w = \frac{\partial F}{\partial z} = \frac{\partial \phi}{\partial u}\cdot\frac{\partial u}{\partial z} + \frac{\partial \phi}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial \phi}{\partial w}\cdot\frac{\partial w}{\partial z}. \tag{7}$$

The solution of the system (2), (3), (4) for x, y, z, and of the system (5), (6), (7) for $\partial\phi/\partial u$, $\partial\phi/\partial v$, $\partial\phi/\partial w$ gives

$$\frac{\partial \phi}{\partial u}\Big/x = 1/(k-1), \qquad \frac{\partial \phi}{\partial v}\Big/y = 1/(k-1), \qquad \frac{\partial \phi}{\partial w}\Big/z = 1/(k-1).$$

Whence,

$$\frac{\partial \phi}{\partial u}\Big/x = \frac{\partial \phi}{\partial v}\Big/y = \frac{\partial \phi}{\partial w}\Big/z.$$

2823 [1920, 185]. Proposed by S. A. COREY, Des Moines, Iowa.

Let TQ and PR be diameters of a circle with center O. Bisect TO at X and draw PQ. On PQ erect the perpendicular XW and on PR, the perpendicular QV. Prove that $OX \cdot PV = PW \cdot PQ$

SOLUTION BY EMMA M. GIBSON, Springfield, Mo.

If S is the intersection of XW and PO, the triangle XOS is isosceles (the reader is requested to draw the figure); for,

$$\angle SXO = 90° - \angle XQW = 90° - \angle SPW = \angle PSW = \angle XSO.$$

Hence $OX = OS = SP$. Since the triangles PWS and PVQ are similar,

$$PW/PV = PS/PQ = OX/PQ,$$

and hence

$$OX \cdot PV = PW \cdot PQ.$$

Also solved by H. L. AGARD, T. M. BLAKSLEE, H. N. CARLETON, P. J. DA CUNHA, H. H. DOWNING, R. M. GINNINGS, C. E. MANGE, H. L. OLSON, LOUIS O'SHAUGHNESSY, ARTHUR PELLETIER, J. B. REYNOLDS, A. V. RICHARDSON, and J. H. WEAVER.

2838 [1920, 273–274].

"A rope is supposed to be hung over a wheel fixed to the roof of a building; at one end of the rope a weight is fixed, which exactly counterbalances a monkey which is hanging on to the other end. Suppose that the monkey begins to climb the rope, what will be the result?"

This problem was invented by Lewis Carroll in December, 1893 (S. D. Collingwood, *The Life and Letters of Lewis Carroll* (*Rev. C. L. Dodgson*), New York, 1899, pp. 317–318), and in his diary he remarked: "Got Professor Clifton's answer [R. B. Clifton, professor of physics at Oxford] to the 'Monkey and Weight Problem.' It is very curious, the different views taken by good mathematicians. Price [Bartholomew Price, professor of physics at Oxford] says that the weight goes *up* with increasing velocity; Clifton (and Harcourt [A. G. Vernon-Harcourt, professor of chemistry at Oxford]) that it goes *up*, at the same rate as the monkey; while Sampson [probably E. F. Sampson, lecturer, tutor and censor of Christ Church, Oxford] says that it goes *down*." Yet another solution by Rev. A. Brook is given on page 268 of *The Lewis Carroll Picture Book* . . . edited by S. D. Collingwood (London, 1899), namely, that "the weight remains stationary."

The problem has been recently discussed in *School Science and Mathematics*, volume 17, December, 1917, p. 821; volume 19, December, 1919, p. 815; and volume 20, February, 1920, pp. 172–173. The editors of the MONTHLY invite mathematical solutions of the problem.

The following solutions were contributed by request:

I. SOLUTION BY E. V. HUNTINGTON, Harvard University.

Case 1. If we neglect the weight of the pulley and rope, the solution follows immediately from the fundamental principle of mechanics, namely: the acceleration of a particle in any direction is proportional to the net force acting on the particle in that direction.

Here there are two particles to consider: (1) the monkey, and (2) the counterpoise.

The net upward force acting on the monkey is $T - W$, where W is the weight of the monkey, and T the tension in his part of the rope. The net upward force acting on the counterpoise is $T' - W'$, where W' is the weight of the counterpoise, and T' the tension in that part of the rope. But on the hypothesis of Case 1, the tension in the rope is the same at all points, so that $T' = T$; also, W' is known to be equal to W. Hence the net upward force acting on the monkey is the same as the net upward force acting on the counterweight, so that *the accelerations of the two bodies must be equal at every instant.*

Therefore, since the two bodies may be supposed to start from rest at the same level, their motions will be precisely parallel, no matter how fast or slow the monkey may climb, up or down, or how much he may allow the rope to slip through his hands.

Case 2. If we take into account the weight of the wheel (still neglecting the weight of the rope), we shall need to use also the equation of rotation.

Let w_0 be the weight of the wheel, r its radius, and k its radius of gyration. Then the equations of motion for the three bodies, W, W', and w_0, will be:

$$T - W = (W/g)dv/dt, \qquad T' - W' = (W'/g)dv'/dt,$$

and

$$Tr - T'r = (w_0/g)k^2 d\omega/dt,$$

where v and v' are the velocities of the monkey and the counterpoise, respectively, in the upward direction in space, and ω is the angular velocity of the wheel.

The geometric conditions of the problem tell us that as long as the rope does not become slack

$$v' = r\omega, \qquad \text{and} \qquad v' + v - u = 0,$$

where u is the relative velocity of the monkey up the rope.

From these five equations, we readily find:

$$\left(W + W' + \frac{w_0 k^2}{r^2}\right)\frac{dv'}{dt} = W\frac{du}{dt} + (W - W')g,$$

which gives the required acceleration, dv'/dt, of the counterpoise, when the relative acceleration, du/dt, of the monkey with respect to the rope is known.

Integrating twice, and putting $W = W'$, we find that the distance x risen by the counterpoise, when the monkey has climbed up a length s on the rope, is

$$x = s/\left(2 + \frac{w_0 k^2}{W r^2}\right),$$

which reduces, as it should, to $x = s/2$ when $w_0 = 0$.

II. Solution by L. M. Hoskins, Stanford University.

If the problem is idealized by neglecting all friction and assuming the cord to be perfectly flexible and without mass, the solution is simple; the further assumption that the pulley is without mass simplifies it still further. The solution of this ideal problem does not furnish a satisfactory answer to the question, what will actually be the result if the monkey begins to climb the rope, for the result may depend in an important way upon the neglected factors. It is, however, useful to consider the simple idealized problem as a preliminary to a more general discussion taking account of friction and the mass of the rope.

(*a*) *Solution Neglecting Friction and Assuming the Cord and Pulley to be Without Mass.*—The initial condition is one in which both the monkey and the counterweight are at rest, each being in equilibrium under the action of two equal and opposite forces,—its own weight and the supporting pull exerted by the cord. In this initial condition the monkey is exerting a downward pull upon the cord equal to his weight; in beginning to climb he increases his pull in order to make the equal and opposite reacting pull exerted upon him greater than his weight, thus giving his center of mass an upward acceleration. Since the cord exerts upon the counterweight a pull equal to that exerted upon the monkey, the two bodies will have equal upward accelerations at every instant; and since both are initially at rest, they will always have equal velocities, and will move equal distances in any time. This agrees with the answer attributed to Clifton and Harcourt.

(*b*) *Effect of the Inertia of the Pulley.*—Let the same assumptions be made as in (*a*) except that the inertia of the pulley is not neglected. Let m denote the mass of the pulley, r its radius, mk^2 its moment of inertia about its axis of rotation, M the mass of the monkey and that of the counterweight, T and T_1 the upward pulls exerted by the cord on the counterweight and monkey respectively, a and a_1 their upward accelerations; the angular acceleration of the pulley will be a/r (assuming that the cord does not slip on the pulley). The following dynamical equations may be written:

For the monkey,

$$Ma_1 = T_1 - Mg;$$

for the counterweight,

$$Ma = T - Mg;$$

for the pulley,

$$mk^2 a/r = (T_1 - T)r.$$

From these equations, $T_1 - T = M(a_1 - a) = mk^2 a/r^2$; $a_1 = a(1 + mk^2/Mr^2)$. The accelerations of the two bodies are thus in a constant ratio, and the velocities acquired and distances described in any time, starting from rest, will be in the same constant ratio as the accelerations.

(*c*) *Effects of Friction and Weight of Cord.*—The weight of the cord causes the tension to vary between the pulley and each suspended body; let the values T, T_1 now refer to the points of tangency of the cord and pulley. Let F denote the difference between T and T_1 which will just maintain uniform rotation of the pulley; then F may be taken as a measure of the friction (including rigidity of the cord). Since F may vary with the velocity, let F_0 be its value for zero velocity (incipient motion). Initially the system is assumed to be at rest with $T = T_1 = Mg + W$, the last term representing the weight of cord below the pulley on each side; at the point of attachment of each of the suspended bodies the tension has the value Mg. In order to begin to climb (*i.e.*, to give to his center of mass an upward acceleration) the monkey has only to exert a pull greater than Mg; unless the pull exceeds Mg by more than F_0, $T_1 - T$ will not become greater than F_0, so that the pulley, cord and counterweight will remain at rest. The monkey will, however, move upward with increasing velocity so long as he maintains a pull which is greater than Mg by any amount whatever, and his velocity will continue undiminished even if the pull becomes equal to

Mg. If, therefore, the monkey wishes to climb the rope without disturbing the counterweight, he can do so if he is skillful enough to exert a pull greater than Mg but less than $Mg + F_0$. This conclusion is not affected by the inertia of the cord and pulley, nor by the weight of the cord.

It is conceivable, however, that the monkey might make such an exertion as to increase the tension by more than F_0; $T_1 - T$ would then become greater than F_0, so that the pulley would have an angular acceleration and the counterweight an upward acceleration. As soon as the counterweight has acquired any velocity it will continue to ascend with undiminished velocity so long as $T_1 - T$ is not less than F. The pull which the monkey would need to maintain to produce this result would be $Mg + F$ if the gravity forces remained balanced; actually the weight of the cord would immediately become unbalanced so that the requisite pull would be less than $Mg + F$ by an amount proportional to the length of cord that has passed the pulley (assuming that the free end of the rope does not reach the floor or other support). If, therefore, the object of the monkey is to raise the counterweight, he can accomplish it if he is able to exert a pull slightly greater than $Mg + F_0$ in order to start the motion. If he should relax his effort immediately, the system might be brought to rest by friction so that the effort would need to be repeated; but after the motion of the cord has proceeded far enough so that the unbalanced weight of the cord reaches the value F so as to overcome friction, the motion will continue without further effort on the part of the monkey. In this case if he wished to check the motion of the counterweight he would need to permit himself to have a downward acceleration; *i.e.*, he would have to produce the requisite decrease in the tension either by very active downward climbing or by letting go of the rope.

(*d*) *General Algebraic Solution.*—The dynamical basis of the foregoing discussion may be embodied in an equation. One method of procedure would be to write separate dynamical equations for the several bodies making up the system and then eliminate the internal forces, as was done above in (*b*). The final equation free from internal forces may, however, be written immediately by applying d'Alembert's principle to the entire system consisting of the monkey, counterweight, cord and pulley. Taking the axis of rotation of the pulley as axis of moments, let the sum of the moments of the mass-accelerations be equated to the sum of the moments of the external forces.

Let x denote the distance of the counterweight above its initial position at any instant, m' the total mass of the cord and ρ its mass per unit length; other notation being as above.

The moment of the weight of the rope is $2\rho gxr$, while the sum of the moments of all other gravity forces is zero, and the moment of the friction is $-Fr$. The only other external force is the normal axle pressure, the moment of which is zero.

The moments of the mass-accelerations are as follows: For the monkey, counterweight, and cord, $-Ma_1r$, Mar, and $m'ar$ respectively; for the pulley, mk^2a/r.

The equation is therefore

$$(M + m' + mk^2/r^2)ar - Ma_1r = 2\rho gxr - Fr,$$

or

$$a = \frac{2\rho gx + Ma_1 - F}{M + m' + mk^2/r^2}.$$

This equation holds when the counterweight is actually rising. Initially $x = 0$ and a will remain zero unless Ma_1 becomes greater than F_0; until this occurs F is to be regarded as representing the actual friction (less than the limiting value F_0) and is equal to Ma_1, so that the equation gives $a = 0$ as it should. After the counterweight has moved from its initial position so that x is no longer 0, a will have a positive value so long as $Ma_1 > F - 2\rho gx$. If $2\rho gx$ becomes $> F$, a will remain positive even if a_1 becomes 0. The equation shows, however, that a can be made negative by giving a_1 a sufficiently great negative value. The conclusions stated under (*c*) are in fact all implied by the above general equation.

The equation also covers the ideal cases (*a*) and (*b*); in the latter it is assumed that ρ, m' and F are zero, while in (*a*) m is also assumed to be zero.

Since $a = d^2x/dt^2$, the general equation is a differential equation of the second order, a_1 being a function of t which depends upon the activity of the monkey, while F is an unknown function of the velocity. If F is treated as constant the equation becomes linear, and may be solved if a_1 is constant or a known function of t.

Thus if, starting from the original balanced condition, a_1 keeps a constant value a_1' for s

certain time, the solution for this part of the motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

If, at the instant $t = t_1$, a_1 changes to the constant value a_1'', the solution for the ensuing motion is

$$x = \frac{Ma_1' - F}{2\rho g}\left[\cosh t\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right] + M\frac{(a_1'' - a_1')}{2\rho g}\left[\cosh (t - t_1)\sqrt{\frac{2\rho g}{M + m' + mk^2/r^2}} - 1\right].$$

Professor Huntington's case 1 was also solved by C. C. WYLIE.

2863 [1920, 482]. Proposed by A. A. BENNETT, University of Texas.

From their generation as roulette curves, show that the two hypocycloids of five cusps drawn with common vertices, are such that each is the envelope of a chord of constant length suitably placed upon the other.

Show that for any odd prime p, the $(p - 1)/2$ distinct p-cusped hypocycloids with common vertices may be arranged in cycles, so that each is the envelope of a chord of constant length taken upon the succeeding curve of the cycle.

A solution of this problem appears on pages 371–373 of this issue of the MONTHLY.

NOTES AND NEWS.

It is to be hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to H. P. MANNING, Brown University, Providence, R. I.

Miss GERTRUDE I. MCCAIN, of Oxford College, has been made professor of mathematics at Westminster College, New Wilmington, Pa.

Associate Professor J. V. MCKELVEY decided to remain at Iowa State College (compare *1921*, 285).

We are requested to state that our note regarding Associate Professor W. A. WILSON's promotion (*1921*, 332) is not in accordance with fact.

At the University of Michigan, Mr. J. P. BALLANTINE, of Pennsylvania State College, and Mr. W. M. COATES, of the University of Virginia, have been appointed instructors of mathematics.

Mr. J. C. FUNK, of Tamalpais Polytechnic High School, Mill Valley, Cal., has been elected to the headship of the department of mathematics in the Santa Maria high school and junior college.

Associate Professor EMMA L. KONANTZ, who has spent two years on leave of absence teaching in Peking University, has returned to her position at Ohio Wesleyan University.

Mr. H. K. CUMMINGS, instructor of physics at the Worcester Polytechnic Institute, and experimental physicist in the research laboratories of the Acheson Graphite Company, has been appointed instructor of mathematics at Brown University.

Mr. F. W. WINTERS, of Mount Allison University, but recently a student instructor at Yale University, has been appointed assistant professor of mathematics at Dalhousie College, Halifax, Nova Scotia.

Assistant Professor J. D. BOND, of the Agricultural and Mechanical College, College Station, Texas, has been appointed associate professor of mathematics in the Louisiana State University.

Mrs. ETHELWYNN R. BECKWITH, formerly assistant professor of mathematics in the College for Women, Western Reserve University, has been appointed acting assistant professor of mathematics in Vassar College.

Dr. F. R. MORRIS, for three years instructor of mathematics in the University of California, has been appointed head of the department of mathematics in the State College of Fresno, which has been recently formed from the normal school and junior college, and which is working under the supervision of the university.

Publications of the Astronomical Society of the Pacific announces that Dr. H. M. JEFFERS, of the University of California, fellow at the Lick Observatory during 1920, passed examinations for the doctorate last January, and has since that time been instructor of mathematics and astronomy at the University of Iowa.

Mr. C. P. ROCKWELL, who studied pure and applied mathematics at the University of Texas, and after graduating at New York University was employed two years in the actuarial department of one of the large insurance companies, has been reappointed state actuary of the Texas department of insurance and banking.

At the University of Saskatchewan, Saskatoon, Professor G. H. LING has resumed his duties as professor of mathematics and dean of the Faculty of Arts [see *1920*, 437]. Dr. L. L. DINES, who has been a junior professor of mathematics [*1919*, 84; also *1914*, 343], was on July 1, 1921, promoted to a senior professorship. He has also been granted leave of absence for the second half of the academic year 1921–1922.

At Purdue University, Professor WILLIAM MARSHALL has been made acting head of the department of mathematics. Instructors C. S. DOAN and F. H. HODGE have been promoted to assistant professorships and Messrs. J. W. BRANSON, E. G. KELLER, J. J. KNOX, and J. H. SHOCK have been appointed to instructorships. The mathematics staff now consists of twenty-one men. Approximately eighteen hundred students are taking mathematics.

THEODORE LYMAN, who (except for service with the A. E. F.) has been teaching at Harvard University since 1902, has been appointed professor of mathematics and natural philosophy. His immediate predecessors in this chair were B. O. PEIRCE, 1888–1914, and W. C. SABINE, 1914–1919. Dr. Lyman has been director of the Jefferson Physical Laboratory since 1910, and was appointed professor of physics in 1917.

In the department of mathematics at the U. S. Naval Academy, Assistant Professor H. M. ROBERT, Jr., has been promoted to an associate professorship and Instructors G. F. ABRICH, R. P. JOHNSON, R. C. LAMB, E. S. MAYER, and J. B. SCARBOROUGH have been promoted to assistant professorships. Dr. G. H. CRESSE (cf. *1919*, 276, 420) has resigned from the staff to accept an associate professorship at the University of Arizona, and Mr. C. D. GREGORY, formerly

of the faculty at Baltimore Polytechnic Institute, and last year a graduate student at the Johns Hopkins University, has been appointed to an instructorship. The 1975 midshipmen in the three classes that receive instruction in mathematics at the Academy are handled in 140 sections.

It was announced in *Science* that R. A. MILLIKAN, professor of physics at the University of Chicago, has been appointed director of the Norman Bridge Laboratory of Physics at the California Institute of Technology, Pasadena, and chairman of the Executive Council of the Institute. He will commence his duties immediately. In order to supplement work in mathematical physics now carried on there by Professor HARRY BATEMAN, Professor H. A. LORENTZ, of the University of Leyden, will be a lecturer and research associate for two months next winter, and Dr. C. G. DARWIN, fellow and lecturer of Christ's College, Cambridge, has been appointed professor of mathematical physics at the Institute for the year 1922–1923.

Dr. HENRI LEBESGUE, recently professor at the University of Paris, has been appointed as successor to HUMBERT (*1921*, 237), professor of mathematics in the Collège de France.

HOWARD ROBERTSON PARK, who became a member of the Association in 1918 (*1918*, 381), died November 12, 1919. He was born at Mount Meigs, Ala., June 28, 1891, and graduated A.B., 1910, from Southern University, where he taught during his senior year. He attended the University of Texas, 1911–12, and summer sessions of the University of Chicago, 1916–1918. He taught in high schools of Texas, New Mexico and California before his appointment in September, 1917, as head of the department of mathematics in the Polytechnic High School, Riverside, Cal. This position he held at the time of his death.

ANNA IRWIN YOUNG, for twenty years head of the department of mathematics in Agnes Scott College, Decatur, Ga., and a charter member of our Association, died September 3, 1920. She was born in Bloom Township (now Chicago Heights), Illinois, November 25, 1873. As a student she attended Westminster College, Pa., 1892–1893; Agnes Scott Institute, Atlanta, 1893–1895; and the summer school of the University of Chicago in 1898 and 1901. In 1898 she received the degree of A.B. from Agnes Scott College, and in 1914 the degree of A.M. from Columbia University. She taught during two summers in the summer school of the University of Georgia.

GEORGE WENTWORTH, one of the authors of the Wentworth-Smith Mathematical Series, died suddenly of heart disease on August 26, 1921. He was born at Exeter, N. H., on January 8, 1868, and was the son of GEORGE A. WENTWORTH (1835–1906), author of the well-known textbooks which bore the Wentworth name, and who for a long time was at the head of the mathematics department of the Phillips Exeter Academy. George Wentworth prepared for college at Exeter and entered Harvard University. In his senior year he left college and took up commercial work in the West, but he later returned to join his father in the writing of mathematical textbooks. He developed great ability in the technique

of this work, and on his father's death he purchased the Wentworth series. In 1913 he and Professor D. E. Smith entered into partnership and founded the Wentworth-Smith Series.

Rev. JOHN BASCOMBE LOCK, fellow and bursar of Gonville and Caius College, Cambridge, England, died September 8, 1921. Born March 18, 1849, he was third wrangler in the mathematical tripos of 1872, assistant master at Eton, 1872–1884, lecturer on mathematics and tutor at Caius College, 1884–1889. He was the author of many well-known texts in arithmetic, geometry, trigonometry, and mechanics.

Colonel JOHN HERSCHEL, youngest son of Sir JOHN F. W. HERSCHEL, died on May 31, 1921, and he was buried at Upton church where lie the remains of his grandfather WILLIAM HERSCHEL; his father was buried at Westminster Abbey. He was born at Cape of Good Hope, October 29, 1837. From 1859 to 1886 he was connected with the Trigonometrical Survey of India, and the Royal Society employed him to observe spectroscopically the eclipses of the sun in 1868 and 1871. He also observed many of the southern nebulæ with the same instruments.

Dr. EMILE BOREL, professor of calculus of probabilities and mathematical physics at the University of Paris, who has received the degree "doctor honoris causâ" from the University of Dublin, is the first Frenchman to be so honored. A Canadian, two Scots and two Irishmen have received similar degrees, but in each case the award was to a man of letters.

The Smith's Prizes at Cambridge University this year have been awarded to L. A. PARS, Jesus College, for an essay on "The general theorem of relativity," and to W. M. H. GREAVES, St. John's College, for an essay on "Periodic orbits in the problem of three bodies." These two annual prizes, now amounting to about £ 23 each, were founded by Robert Smith (1689–1768), master of Trinity College, elected in 1716 to succeed Cotes as Plumian professor of astronomy at Cambridge. They have been awarded annually, since 1769, except in 1884, to "two commencing Bachelors of Arts, the best proficients in mathematics and natural philosophy," determined by essays of greatest merit. New regulations were promulgated in 1884, and again in 1909 when Raleigh prizes were awarded on a similar basis. H. BATEMAN was a Smith's prizeman for 1905, J. H. JEANS for 1901, and R. C. MACLAURIN for 1897.

On May 17, 1921, Dr. O. D. KELLOGG, associate professor of mathematics at Harvard University, lectured before the Providence Engineering Society on "Submarine listening devices."

Professors G. D. BIRKHOFF and L. E. DICKSON are to lecture in the summer school of the University of California in 1922.

We have recently made numerous references to the activities of Professor SOLOMON LEFSCHETZ, of the University of Kansas (*1920*, 339, 340, 436, 440; *1921*, 241, 288, 384). His lectures at the University of Rome, in April and early

May, 1921, were on "Analysis situs and algebraic geometry," and are to form the basis of a monograph due to appear in the Borel series.

Among the papers read at the twenty-sixth meeting of the American Astronomical Society, held at the Van Vleck Observatory, Middletown, Conn., August 30–September 1, 1921, were the following: "New measures of solar activity and the 'earth-effect'" by L. A. BAUER; "A theory for the Trojan group of asteroids" by E. W. BROWN; "Gilbert's bombardment hypothesis" by J. L. COOLIDGE; "Parallaxes of 65 stars" by J. A. MILLER.

At meetings of the Royal Society of Canada, mathematical, physical and chemical section, held at Ottawa, May 18–20, 1921, the papers presented included the following: "Division in relation to the algebraic numbers," presidential address by J. C. FIELDS; "On the reduction of the circulants to polynomial form with applications to the circulants of the 7th and 11th degrees" by J. C, GLASHAN; "The gravitation potential of an anchor ring; some tidal problems" by A. H. S. GILLSON; "The solution of plane triangles by nomographic charts" by S. D. KILLAM; "Note on the geometrical equivalent of certain invariants" by C. T. SULLIVAN.

On October 24, 1921, a bill to incorporate the American Mathematical Society in the District of Columbia was introduced into the Senate of the United States by Mr. Lodge. It was twice read and referred to the committee of the judiciary. —*Congressional Record*, October 24, 1921.

In order to provide an enduring memorial for the 127 Field Service men who lost their lives in the great war, American Field Service Fellowships for French Universities were established (compare *1921*, 44). The fellowships for 1922–23, not to exceed 25 in number, are tenable for one year by citizens of the United States, and are to be of the value of $200 plus 10,000 francs. Under certain circumstances the fellowships are renewable a second year. The 31 fields of study possible for fellows include Mathematics, Engineering, and Astronomy. Prospective applicants for these fellowships should communicate with the secretary of the Trustees, Dr. I. L. Kandel, 522 Fifth Avenue, New York City. The next awards are to be made early in 1922.

The University of Rome has become one of the great mathematical centers of the world through the following recent appointments to its chairs: FEDERIGO ENRIQUES, of the University of Bologna, FRANCESCO SEVERI, of the University of Padua, and GIUSEPPE BAGNERA, of the University of Palermo. VITO VOLTERRA, GUIDO CASTELNUOVO, and TULLIO LEVI-CIVITA, were already professors there.

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THE APRIL MEETING OF THE IOWA SECTION.

The eighth regular meeting of the Iowa Section was held, in conjunction with the twenty-fifth annual meeting of the Iowa Academy of Science, at Simpson College, Indianola, on April 30, 1921.

The attendance was seventeen, including the following fifteen members of the Association:

Julia T. Colpitts, C. W. Emmons, M. E. Graber, E. C. Kiefer, G. E. King, F. M. McGaw, J. V. McKelvey, E. A. Pattengill, J. F. Reilly, H. L. Rietz, Maria M. Roberts, W. J. Rusk, B. F. Simonson, F. M. Weida, C. W. Wester.

The following officers were elected for the ensuing year: W. J. RUSK, Chairman; C. W. EMMONS, Vice-chairman; and J. F. REILLY, Secretary-treasurer. It was voted to accept the proposal of the Iowa Academy of Science that the Chairman of the Iowa Section become a member of the executive committee of the Academy, with a view to a closer coöperation of the two bodies. In accepting the proposal it was understood that the Iowa Section's independence, or relation to the Association, was in no way impaired.

Professor McKelvey's committee on the reports of the National Committee on Mathematical Requirements was continued.

The following papers were presented:

(1) "Correlation between mental tests and grades in mathematics of freshman engineering students" by Professor MARIA M. ROBERTS, Iowa State College;

(2) "Playing with the sine and projection formulas" by Professor W. J. RUSK, Grinnell College;

(3) "Certain summation formulas" by Professor J. F. REILLY, State University of Iowa;

(4) "Some properties of the function $w = \tanh z$" by Mr. F. M. WEIDA, State University of Iowa;

(5) "Derived solutions of differential equations" by Professor M. E. GRABER, Morningside College;

(6) "The surface $z = \log_y x$" by Professor C. W. EMMONS, Simpson College;

(7) "Circles mutually tangent and tangent to concentric circles at specific points" by Professor C. W. WESTER, Iowa State Teachers College;

(8) "A study of certain reports of the National Committee on Mathematical Requirements." Committee report by Professor J. V. MCKELVEY, Chairman.

Abstracts of papers follow below, the numbers corresponding to the numbers in the list of titles:

1. Professor Roberts reported on a study made of the college records of 337 freshman engineers who took the Thurstone engineering intelligence test. Correlations were found, first, between college mathematics and the first four tests, arithmetic, algebra, geometry and general intelligence; second, between college mathematics and the first three, arithmetic, algebra and geometry; and third,